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MATHEMATICS

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ELEMENTS
OF
GEOMETRY;

WITH THEIR APPLICATION TO THE
MENSURATION OF SUPERFICES AND SOLIDS,
TO THE DETERMINATION OF THE
MAXIMA AND MINIMA OF GEOMETRICAL QUANTITIES,
AND TO THE CONSTRUCTION OF A GREAT VARIETY OF
GEOMETRICAL PROBLEMS.

By **THOMAS SIMPSON, F.R.S.**

AND PROFESSOR OF MATHEMATICS, AT THE ROYAL MILITARY ACADEMY, WOOLWICH.

A NEW EDITION,

CAREFULLY CORRECTED—WITH ADDITIONAL NOTES AND AN APPENDIX, CONTAIN-
ING A DESCRIPTION OF THE ANALYTICAL AND SYNTHETICAL MODES OF
REASONING MADE USE OF BY MATHEMATICIANS—AN ACCOUNT
OF THE LOST WRITINGS OF EUCLID AND APOLLONIUS,
AND OF THE SEVERAL ATTEMPTS OF THE
MODERNS TO RESTORE
THEM.

LONDON:

PRINTED FOR J. COLLINGWOOD, IN THE STRAND.

1821.

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TO THE
HONORABLE
CHARLES FREDERICK, Esq.

SURVEYOR-GENERAL OF HIS MAJESTY'S ORDNANCE,
&c. &c. &c.

HONORABLE SIR ;

THE subject of the sheets which I here beg leave to lay before You, is of so much consequence to mankind, as justly to claim the regard and sanction of the Great. Geometry is, not only a most accurate, but a very extensive science, whose application and great utility, as well in the arts of peace as of war, are well known to You.

But though this work, if the manner in which it is executed be correspondent to the importance of the subject, may not want sufficient merit to render it worthy of the approbation of a Gentleman, who, amidst a multiplicity of public employments, preserves an undiminished ardour for the sciences and a knowledge of the works of art and nature ; yet I have, Sir, still farther motives for this address : Your

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DEDICATION.

great influence and zeal to promote the good of an institution under which I am placed, and the favours that I have received at your hands, make me earnest to embrace this opportunity of testifying publicly, that I am,

HONORABLE SIR,

With great respect,

Your much obliged,

and most obedient

humble servant,

THOMAS SIMPSON.

Royal Academy,
Woolwich,
March 3, 1760.

P R E F A C E.

MY design in writing upon the subject of Geometry, was to open an easy way for young beginners to arrive at a proficiency in that useful science; without either being obliged to go through a number of unnecessary propositions, or having recourse to the *ungeometrical* methods of demonstration, that abound in most modern compositions of this nature.

The difficulty of the undertaking, I was not unapprised of; and objections occurred that were not easy to be removed: Nevertheless, I have grounds to hope, from the reception my first attempt has met with, that my endeavours have not been entirely unsuccessful. No pains have, indeed, been spared, to render the work useful: And I flatter myself, that the spirit and rigour of demonstration, so essential to the subject, are also tolerably well preserved; though I have not been so intent to guard against the attack of *Critics*, as to lose sight of my main design of furnishing a plain, easy institution for learners: Yet I have strong hopes, that there will not be found in these sheets, any inaccuracies, or oversights, that are absolutely unpardonable. To expect a faultless piece is impossible: And I well know, that the most elaborate and best approved systems of Geometry extant, are not without many imperfections. But, were the smallest imperfection to be a real fault, my ambition would rather be, to show some degree of judgment, by avoiding a multitude of such faults, than by exposing and magnifying the flaws of other writers. It is more easy to see a fault, than to avoid one: And those men who are the most sanguine to distinguish themselves at the expense of others, are generally observed to stand in need of greater indulgencies, than even the persons whom they unmercifully attack. But I shall put an end to this digression by pointing out one objection, that may be

brought against this work ; which is, that in demonstrations admitting of several cases, the most easy ones are sometimes omitted ; and that the converse of some propositions is not at all demonstrated. But this, I conceive, will be found a real advantage to the learner ; without which, it would have been impossible to have comprised the Elements in the compass they now take up. Besides, the greatest part of the demonstrations omitted being such as may be inferred from those given, by means of Axioms only ; they may, therefore, be easily supplied by any reader, should they happen to become necessary, which I have scarce ever found to be the case. But, even allowing this to be a defect, it is abundantly compensated by the extensive application given in the three last sections ; which is infinitely more useful, in itself, and more necessary to the forming an able *Geometrician*, than any thing of the kind we have been speaking of.

In this second edition, (which is, in a manner, a new work,) many considerable alterations and additions have been made. The order of some of the first propositions is changed : And some difficult propositions in the second book are rendered more plain. In the fourth book several new Theorems on proportions are added. The solid Geometry is now connected with the *plane*, and is demonstrated with the same accuracy. The mensuration of Superficies and Solids is also more explicitly handled ; and the demonstration of the several rules is here established on a better foundation, than even in authors who have wrote professedly on the subject. The *Maxima* and *Minima*, and the construction of Geometrical Problems, are likewise considerably extended and improved. And, at the end, Notes geometrical and critical, very useful to improve the judgment of young students, are now added.

ADVERTISEMENT TO THIS NEW EDITION.

THE Public are here presented with a new Edition of the celebrated THOMAS SIMPSON'S ELEMENTS OF GEOMETRY—a work, which, for upwards of seventy years, has been considered one of the easiest, and at the same time one of the most logical and comprehensive Treatises on the Subject, either in the English, or perhaps in any other Language; and, notwithstanding the time that has elapsed since its first appearance, it will still bear a comparison with any modern production on the same subject; many Authors, indeed, both domestic and foreign, have copied largely from it, without acknowledging to whom they were indebted:—Bossut, L'Huillier, and Le Gendre, for instance, have made very free with his Section on the MAXIMA and MINIMA of Geometrical Quantities, which is the more inexcusable, as our Author was the first who introduced that subject into the Elements, of which circumstance the writers abovementioned can hardly be supposed to have been ignorant; and the value of his extensive collection of Problems, has been long known and appreciated by every Geometer who has accustomed himself much to the solution of Problems; the constructions in this book having always been considered models of Geometrical purity and elegance.

With respect to this new Edition—the Editor has only to add, that it has been carefully corrected; several Notes, and references to other books have also been added, particularly among the problems, which, it is hoped, will be found useful to the inquisitive reader, inasmuch as they will enable him at once to consult other authors, where the same subjects are often differently, though certainly not more elegantly treated. A few Theorems, (marked A. B. &c.) have been restored from the early Editions; and two remarkable properties of the circle have been taken from Apollonius on Plane Loci, the second of which (B. page 84,) will be found of very frequent use in the Construction of Triangles.

In the APPENDIX, a somewhat detailed account is given of that beautiful mode of investigation of which the Ancients were such mighty masters, and of such of the Analytical Writings of EUCLID

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and APOLLONIUS, as belong to *Plane Geometry*, which, unfortunately, have long been lost to Science. Their other works, namely, the *Conics of Apollonius*—the *Solid Loci of Aristæus*—the *Loci in Superficies of Euclid*—and the *Loci ad Medietates of Eratosthenes*, belonging to the *higher Geometry*, could not with propriety be introduced in an elementary work.

To conclude—the Editor feels not a little gratification, in being able to inform the *Mathematical reader*, that new Editions of the scarce and valuable works of the learned author of this volume, with his last additions and corrections, are now in progress, and may, at no distant period, be had of the Publisher, who is the Proprietor of all MR. SIMPSON'S Works.

M. FRYER.

October, 1821.

INTRODUCTION.

INTRODUCTION.

AS in every work of this nature, designed to contain whatever may be most requisite to the forming of a regular and complete system of Geometry, a number of Propositions must necessarily have a place, whose chief use and application lie in the higher branches of the Mathematics; but there being many persons, particularly young gentlemen in public schools, who have occasion to learn so much Geometry only, as is necessary to give them a proper introduction into its practical and most common applications, such as Mensuration, Trigonometry, Navigation, Fortification, Perspective, &c. it may be of service, to point out to such Readers, what Propositions in these Elements may be omitted, as least useful to them; without either hurting the connection, or taking away from the evidence of the other demonstrations. The numbers of these propositions, in the several books, are as follow.

IN BOOK I.

The 6, 17, 19, 21, 22, 23, and 29th.

IN BOOK II.

The 4, 5, 10, 11, 12, 13th, and the 2d Corol. to the 9th.

IN BOOK III.

The 4, 5, 6, 7, 8, 9, 15, 18, 19, 20, 25, 26, 67, and 28th.

IN BOOK IV.

The 4, 5, 6, 9, 11, 13, 16, 17, 20, 21, 22, 23, 25, 26, 27, 28, and 29th.

IN BOOK V.

The 1, 2, 16, 17, 18, 19, 20, 25, 26, 28, and 31st.

IN BOOK VI.

The two or three first propositions only, need be read; except by those who are concerned in surveying and dividing of lands: to whom the whole book will be highly useful.

Also, with regard to the seventh book, if *Perspective* be the only application in view, (which I have known frequently to be the case,) the 1st, 2d, 4th, and 12th propositions may suffice. But if a more general idea of the properties of intersecting planes should be required, such as is necessary in the doctrine of solids and spheric geometry, then all the propositions, to the 12th, ought to be taken.

The 17th, 19th, 20th, 21st, 22d, and 23d propositions of this seventh book should also be read by those who would be desirous of finding the content and proportion of solid bodies; as should, likewise, the whole eighth book; except, perhaps, the first and ninth propositions, together with the three first lemmas; which may be thought too plain, by those who are not very solicitous about geometrical rigour, to need a demonstration.

An INDEX or TABLE referring to the places in these Elements, where all the most material propositions in the first six, and in the eleventh and twelfth books of Euclid, are demonstrated.

<i>Euclid.</i>	<i>These El.</i>	<i>Euclid.</i>	<i>These El.</i>	<i>Euclid.</i>	<i>These El.</i>	<i>Euclid.</i>	<i>These El.</i>
<i>Book I.</i>	<i>P. B.</i>	<i>B. II.</i>	<i>P. B.</i>	<i>B. V.</i>	<i>P. B.</i>	<i>B. XI.</i>	<i>P. B.</i>
<i>Prop. 5</i>	12. 1	<i>P. 11</i>	19. 5	<i>P. 16</i>	2. 4	<i>Pr. 5</i>	{ <i>Cor. to</i>
6	18. 1	12 }	10. 2	17 }		6	{ 2. 7
8	14. 1	13 }		18 }	3. 4	8	4. 7
9	5. 5	14 }	9. 6	19 }		9	5. 7
10	6. 5			22	5. 4	10	8. 7
11	3. 5	<i>B. III.</i>	<i>P. B.</i>	24	6. 4	12	9. 7
12	4. 5	<i>Pr. 3</i>	2. 3	25	3. 4	14	3. 7
13	1. 1	7 }		<i>B. VI.</i>	<i>P. B.</i>	15	7. 7
15	3. 1	8 }	5. 3	<i>Pr. 1</i>	7. 4	16	10. 7
18 }		11	8. 3	2	12. 4	18	11. 7
19 }	13. 1	12	7. 3	3	18. 4	19	6. 7
21	23. 1	14	3. 3	4	14. 4	24	13. 7
22	8. 5	15	4. 3	5	17. 4	25	16. 7
23	7. 5	17	21. 5	6	15. 4	28 }	21. 7
24	21. 1	20	10. 3	7	16. 4	29 }	
26	15. 1	21	11. 3	8	19. 4	30 }	17, 19 and 20
27 }		22	17. 3	9	11. 5	31 }	of the 7 th , and Cor. 3.
28 }	8. 1	25	18. 3	10	15. 5	32 }	and Cor. 8.
29	7. 1	31	16. 3	11	12. 5	33	21. 7
31	9. 5	32	14. 3	12	13. 5	34	9. 8
32 }	9. 1	33	22. 5	13	14. 5	35	23. 7
33	10. 1	34	25. 5	14 }	<i>Cor. to</i>	37 }	14. 7
34	26. 1	35	21. 3	15 }	25. 4		{ 25. 7
35	24. 1	36 }	22. 3	16 }		<i>B. XII.</i>	<i>P. B.</i>
36 }	2. 2	37 }		17 }	10. 4	<i>Pr. 2</i>	3. 8
37 }		<i>B. IV.</i>	<i>P. B.</i>	18	11. 6	5	<i>Cor. 4</i>
38 }	<i>Cor. to</i>	<i>Pr. 2</i>	25. 5	19	24. 4	6 }	to 8. 8.
41 }	2. 2	3	26. 5	20	26. 4	7 }	7. 8
42	6. 6	4	24. 5	22	28. 4	8	9. 8
43	3. 2	5	23. 5	23	25. 4	9 }	
44	6. 6	10 }		25	13. 6	10 }	
45	7. 6	11 }	28. 5	27 }	17. 5	11 }	
46	10. 5	12 }	30. 5	28 }	18. 5	12 }	
47	8. 2	15	29. 5	29	19. 5	13 }	
				30	29. 4	14 }	
				31		15 }	
<i>B. II.</i>	<i>P. B.</i>	<i>B. V.</i>	<i>P. B.</i>	<i>B. XI.</i>	<i>P. B.</i>		
<i>Pr. 1</i>	5. 2	4	4. 4	<i>Pr. 3</i>	1. 7	18	{ <i>Cor. to</i>
4	6. 2	12	6. 4	4	2. 7		{ 11. 8
5	7. 2	15	1. 4				

MATHEMATICAL WORKS

OF THE LATE

THOMAS SIMPSON, F.R.S.

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ELEMENTS

ELEMENTS

OF

GEOMETRY.

BOOK I.

DEFINITIONS.

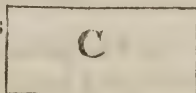
1. **GEOMETRY** is that science, by which we compare such quantities together as have extension.

Extension is distinguished into length, breadth, and thickness.

2. A *Line* is that which has length without breadth.

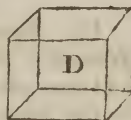
The terms, bounds, or extremes of a Line are points.

3. A *Surface* is that which has length and breadth only, as C.



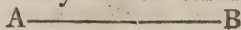
The bounds of a surface are lines.

4. A *Solid* is that, which has length, breadth, and thickness, as D.



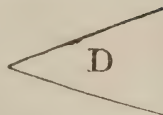
The bounds of a Solid are surfaces.

5. A *Right* (or *straight*) *line* is that, which lies evenly between its extremes, or which every where tends the same way, as AB.

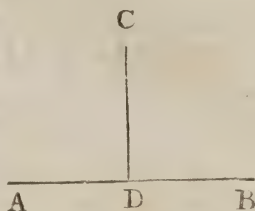


6. A *Plane-surface* is that, which is every where perfectly flat and even, or which touches, in every part, any right-line extended between points any where taken in that surface.

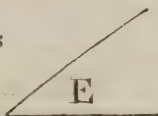
7. An *Angle* is the inclination, or *opening*, of two right-lines meeting in a point, as D.



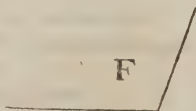
8. When one right line DC, standing upon another AB, makes the angles on both sides equal, those angles are called *right angles*; and that line CD is said to be *perpendicular* to the other, AB, on which it insists.



9. An *Acute-angle* is that, which is less than a right-angle, as E.



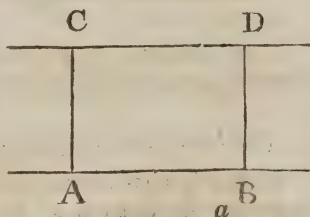
10. An *Obtuse-angle* is that, which is greater than a right-angle, as F.



11. The *distance* of two points, is the right-line reaching from the one to the other.

12. The *distance* of a point from a line, is a right-line drawn from that point, perpendicular to, and terminating in, the line given.

13. *Parallel* (or *equidistant*) *right-lines* AB, CD are such, which being in the same plane-surface, if infinitely produced, would never meet.

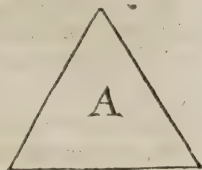


14. A *Figure*, in Geometry, is a *bounded space*, and is either a *surface*, or a *solid*.

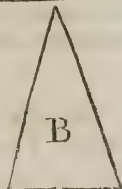
15. A *right-lined plane Figure* is that, formed in a *plane-surface*, whose terms, or bounds, are *right-lines*.

16. All plane Figures bounded by three right-lines, are called *Triangles*.

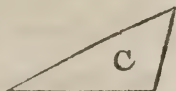
17. An *Equilateral Triangle* is that, whose bounds or sides are all equal, as A.



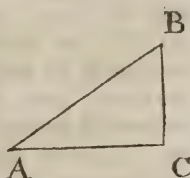
18. An *Isosceles Triangle* is, when two sides are equal, as B.



19. A *Scalene Triangle* is, when all the three sides are unequal, as C.



20. A *Right-angled Triangle* is that which has one right-angle, as ACB; whereof the side AB opposite to the right-angle, is called the *Hypotenuse*.

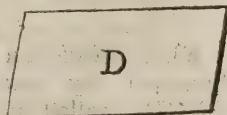


21. An *Obtuse-angled Triangle* is that which has one obtuse angle.

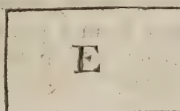
22. An *Acute-angled Triangle* is that which has all its angles acute.

23. Every plane figure bounded by four right lines, is called a *Quadrangle* or *Quadrilateral*.

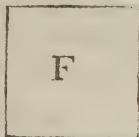
24. Any *Quadrangle*, whose opposite sides are parallel, is called a *Parallelogram*, as D.



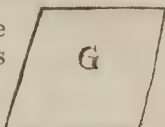
25. A *Parallelogram*, whose angles are all right-ones, is called a *Rectangle*, as E.



26. A Parallelogram whose sides are all equal, and its angles all right ones, is called a *Square*, as F.



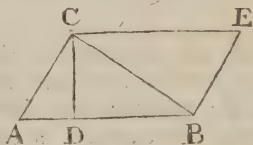
27. A Parallelogram whose sides are all equal, but its angles not right, is called a *Rhombus* as G.



28. All other four-sided figures, besides these, are called *Trapeziums*.

29. A right-line joining any two opposite angles of a four-sided figure, is called a *Diagonal*, as CB.

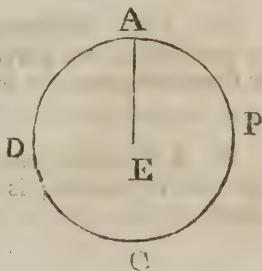
30. That side AB upon which any parallelogram ACEB, or triangle ACB is supposed to stand, is called the *base*; and the perpendicular CD falling thereon from the opposite angle C, is called the *altitude* of the parallelogram, or triangle.



31. All plane figures contained under more than four sides, are called *Polygons*; those having *five sides*, are called *Pentagons*; those having *six sides*, *Hexagons*; and so on.

32. A *Regular Polygon* is one whose angles, as well as sides, are all equal.

33. A *Circle* is a plane figure, bounded by one curve-line APCD, called its *circumference*, every where equally distant from a point E within the circle, which point is called the *center*.



34. The *Radius* of a circle, is the distance of the center from the circumference, or a right-line EA drawn from the center to the circumference.

AXIOMS, OR SELF-EVIDENT TRUTHS.

1. Things, which are equal to one and the same thing, are also equal to each other.

2. Every whole is greater than its part.

3. Every whole is equal to all its parts taken together.

4. If to equal things, equal things be added, the wholes will be equal.

5. If from equal things, equal things be taken away, the remainders will be equal.

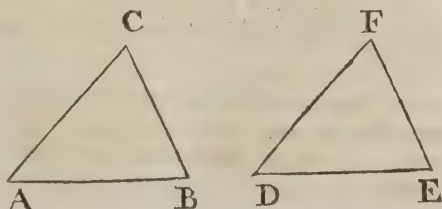
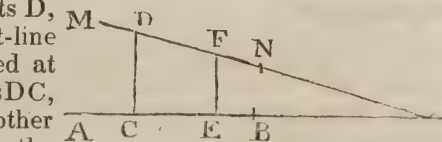
6. If to, or from unequal things, equal things be added, or taken away, the sums, or remainders, will have the same difference, as the unequal things first proposed.

7. All right-angles are equal to one another.

8. More than one right-line cannot be drawn from one given point A to another given point B. A ————— B

9. If two points D, F, in a right-line MN, are posited at unequal distances DC, FE, from another right-line AB in the same plane-surface; those two lines being produced on the side of the least distance EF, will meet each other.

10. If two right-lines CA, CB, making an angle C, be respectively equal to two other right-lines FD, FE, making an



angle F, and the angles which they make C and F, be likewise equal; the right-lines AB, DE joining their extremes will be equal, and the two triangles ACB, DFE equal in all respects.

If this should not appear sufficiently evident for an axiom, conceive the triangle DFE to be removed, and so applied to the triangle ABC, that the point F may coincide with C, and the side FD fall upon the side CA; then, because FD is supposed equal to CA, the point D will also fall upon A. And the angle F being equal to the angle C, the side FE will fall upon CB; and consequently the point E upon the point B, because FE is supposed equal to CB. Therefore, seeing all the bounds of the two triangles coincide, it is manifest, that not only the bases AB, DE, but the angles opposite to the equal sides, are also equal.

When all the four lines CA, CB, FD, FE, are equal; the triangle DFE, being *contrariwise* applied to ACD, so that FE may coincide with CA, will, *also*, agree with the triangle ACB (as is manifest from the reasoning above:) and so, the angle E (as D did before) now coinciding with the angle A, the two angles E and D must necessarily be equal to each other, in this case, where the triangle DFE is an isosceles one.

POSTULATES, OR PETITIONS.

1. That, from any given point, to any other given point, a right-line may be drawn.
2. That a right-line may be produced, or continued out, at pleasure.
3. That, from any point as a center, with a radius equal to any right-line assigned, a circle may be described.
4. That a right-line may be drawn perpendicular to another, at any point assigned; and that it is also possible to make a right-line, or a right-lined angle, equal to the whole, or to the half, of any right-line, or right-lined angle assigned.

This fourth Postulate is added, more for the sake of making the proper references, than through absolute necessity; since, what is here barely assumed as possible, is effected, and actually demonstrated, in the beginning of the Fifth Book, entirely independent of every thing but Axioms and the other Postulates, above laid down. It may also be proper to note here, that, though these Postulates are not always quoted, it will be easy to perceive where, and in what sense, they are to be understood.

NOTES and OBSERVATIONS, with the significations of Terms and Signs, used in this Work.

A PROPOSITION is when something is either proposed to be done, or to be demonstrated, and is either a problem or a theorem.

A PROBLEM is a practical proposition, in which something is proposed to be *done*.

A THEOREM is a speculative proposition, in which something is proposed to be *demonstrated*.

A LEMMA is when some premise is demonstrated, in order to render the thing in hand the more easy.

A COROLLARY is a consequent truth, gained from some preceding truth or demonstration.

A SCHOLIUM is when remarks and observations are made upon something going before.

An HYPOTHESIS is a *supposition* or *assumption*, made either in the enunciation of a proposition, or in the course of a demonstration.

The signification of SIGNS.

The sign $=$, denotes that the quantities betwixt which it stands are equal.

The sign $\sqsupset$, denotes that the quantity preceding it, is greater than that which comes after it.

The sign $\sqsubset$, denotes that the quantity preceding it, is less than that which comes after it.

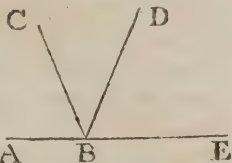
The sign $+$, denotes that the quantity which it precedes is to be added.

The sign $-$, denotes, that the quantity which it precedes, is to be taken away, or subtracted.

A figure, or number, prefixed to any quantity, shows

how often that quantity is to be taken, or repeated ; as 5 A shows, that the quantity represented by A, is to be taken 5 times.

When several angles are formed about the same point (as at B,) each particular angle is described by *three letters*, of which the *middle one* shows the *angular point*, and the other two, the lines that form the angle: thus CBD or DBC signifies the angle formed by the lines CB and DB.



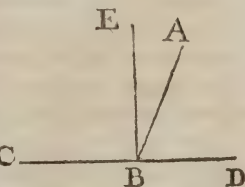
When, in any demonstration, you meet with several quantities joined the one to the other continually by the mark of equality ($=$), the conclusion drawn from thence, is always gathered from the first and last of them ; which are equal to each other, by virtue of the first axiom. Thus if $A=B=C=D$, then will the first (A) and the last (D) be equal to each other.

Also, when in the quotations you meet with two numbers, the first shows the proposition, and the second the book. Moreover, Ax. denotes axiom ; Post. postulatum ; Def. definition ; Hyp. hypothesis. Note also, that, whenever the word *line* occurs, without the addition of either *right*, or *curved*, a right-line is always understood : and that, when a line is said to be drawn to, or from an angle, the angular point is meant.

THEOREM I.

A line (AB) standing upon another line (CD) makes with it two angles (ABC, ABD) which, taken together, are equal to two right-angles.

If the angles ABC, ABD are equal, it is plain they make two right angles^a ; if *unequal*, let BE be perpendicular to CD^b, dividing the greater of them (ABC) into the parts EBC, EBA ; then the former part EBC being a right-angle^a, and



^a Def. 8.

^b Post. 4.

the remaining part EBA together with the whole less angle ABD, equal to another right-angle EBD^c; the whole, of both the proposed angles, taken together, must necessarily be equal to two right-angles^d.

^d Ax. 4.

COROLLARY.

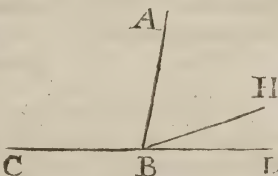
Hence all the angles at the same point (B) on the same side of a right-line (CD) are equal to two right-angles^e.

^e Ax. 3.

THEOREM II.

If one line (AB) meeting two others (BC, BD) in the same point (B), makes two angles with them (ABC, ABD) which together are equal to two right-angles; these lines (BC, BD) will form one continued right-line.

For, if possible, let BH, and not BD, be the continuation of the right line CB; then the angles ABC and ABH being = two right-angles^e = ABC and ABD^f; if from these equal quantities, ABC common to both, be taken away, there will remain ABH = ABD^g; which is impossible^h.



^e 1. 1.

^f Hyp.

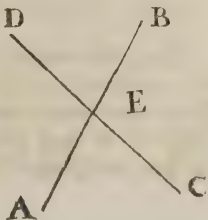
^g Ax. 5.

^h Ax. 2.

THEOREM III.

The opposite angles (DEB, AEC), made by two lines (DC, BA) intersecting each other, are equal.

For $DEB + DEA = \text{two right-angles}^i = AEC + DEA$: whence, by taking away DEA, common, there remains $DEB = AEC^k$.



ⁱ 1. 1.

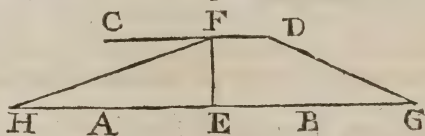
^k Ax. 5.

THEOREM IV.

Two right-lines (AB, CD) perpendicular to one and the same right-line (EF), are parallel to each other.

If you say, they are *not* parallel; then let them, when produced out, meet in some point, as G.

In EA, produced (if necessary) let there be taken $EH = EG$ ^l, and let the right-



^l Post. 4.

^m Post. 1.

ⁿ Def. 8.

^o Ax. 10.

^p Hyp.

^q 2. 1.

^r Ax. 8.

line FH be drawn^m. The triangles EHF and EGF, having $EH = EG$, the angle $HEF = GEF$ ⁿ, and EF common, are therefore equal in all respects^o: and so, the angle EFH being $= EFG$ (EFD) = a right-angle^p, HFDG (as well as HEG) must be one continued right-line^q: which is impossible^r. Therefore AB and CD are parallels.

SCHOLIUM.

In this theorem, the *possibility* of parallel lines (or such, which being infinitely produced, in the same plane, can never meet) is demonstrated: for EF may be drawn perpendicular to AB^l; and CFD, again perpendicular to EF^l; which last, it is demonstrated, will be parallel to AB.

THEOREM V.

Perpendiculars (EF, GH) to one (AB) of two parallel lines (AB, CD) terminated by those lines, are equal to each other; and also perpendicular to the other of the two parallels (CD).

For, AB and CD being parallel to each other, GH can neither be greater nor less than EF^s; and therefore must be equal to EF. If you say, that EF is *not* perpendicular to CD; then let FM be perpendicular to EF^t, meeting GH produced (if

^s Ax. 9.

and Def. 13.

^t Post. 4.

$\approx \Delta x, 2.$

Hence, *through the same point F, more than one parallel cannot be drawn to the same line given AB.*

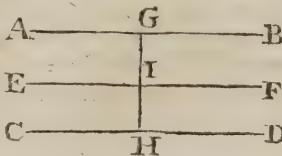
From the preceding proposition, the consistence of the twenty-fifth definition, or the possibility, that all the properties ascribed to a rectangle, can subsist together in the same figure, will appear, together with the method of con-

G	H
E	F

struction. For, at any two points CD, in a right line RS, two perpendiculars, CG, DH may be erected^y; and a perpendicular to R C D S^y Post. 4. one of these, at any point E, meeting the other in F, may be drawn. The figure CEFD thus constructed will be a rectangle: for CE and DF are parallel^z; as^z 4. 1. are also CD and EF^z: therefore the angle F (as well as C, D, and E) is a right-angle^a. If CE be made=^a 5. 1. CD, then will the rectangle CEDF have all its sides equal^b. Which answers to the definition of a square. ^b 5. 1. and Ax. 1.

Right-lines (ΔB , EF) parallel to the same right-line (CD) are parallel to each other.

For let the line HIG be perpendicular to CD : then, that line being also perpendicular to both AB and EF , these last are parallel to each other^d.

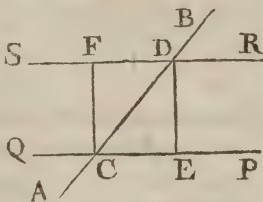


d 4. 1.

THEOREM VII.

A line (AB) intersecting two parallel lines (SR, QP) makes the alternate angles (SDC, PCD) equal to each other.

Let CF and DE be perpendicular to QP, and SR^e; then these lines FC and DE are likewise parallels^f; and so the triangles CFD and CDE, having the side CF=DE^e, FD=CE^e, and the angle F=E^g, they will also have the angle FDC=ECD^h.



COROLLARY I.

Hence, a line intersecting two parallel lines, makes the angles (BDR, BCP) on the same side, equal to each other; for BDR (=CDSⁱ)=BCP^k.

ⁱ 3. 1.
^k 7. 1.

COROLLARY II.

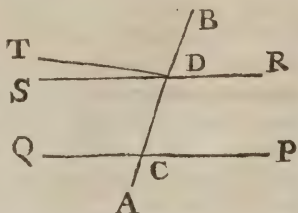
Hence, also, a line falling upon two parallel lines, makes the sum of the two internal angles (SDC+QCD) on the same side of it, equal to two right-angles: for the angle SDC being=PCD, and PCD+QCD=two right angles^l; thence is SDC+QCD=also to two right-angles^m.

^l 1. 1.
^m Ax. 4.

THEOREM VIII.

If a line (AB) intersecting two other lines (PQ, RS), makes the alternate angles (DCP, CDS) equal to each other; then are those two lines parallel.

For, if possible, let some other line DT, and not DS, be parallel to PQⁿ; then must CDT=DCP^o=CDS^p: which is impossible^q.



ⁿ Sch. to 4. 1.
^o 7. 1.
^p Hyp.
^q Ax. 2.

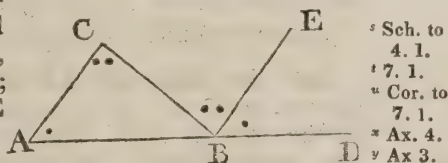
COROLLARY.

Hence, if a line falling on two others, makes the angles (BDR, BCP) above them on the same side, equal to each other; then those two lines are parallel: because $SDC = BDR$. 3. 1.

THEOREM IX.

If one side (AB) of a triangle (ABC) be produced, the external angle (CBD) will be equal to both the internal opposite angles (A, C) taken together.

For, let BE be parallel to AC^s; then will the angle $C = CBE$ ^t, and the angle $A = DBE$ ^u; therefore $C + A = CBE + DBE$ ^x = CBD ^y.



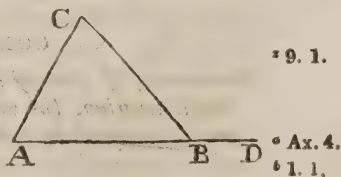
COROLLARY.

Hence the external angle of a triangle is greater than either of the internal, opposite angles.

THEOREM X.

The three angles of any plane triangle (ABC) taken together, are equal to two right-angles.

For, if AB be produced to D, then $C + A = CBD$ ^z, to which equal quantities let the angle CBA be added, then will $C + A + CBA = CBD + CBA$ ^a = two right-angles^b.



COROLLARIES.

1. If two angles in one triangle, be equal to two

* Ax. 5. *angles in another triangle, the remaining angles will also be equal^c.*

2. *If one angle in one triangle, be equal to one angle in another, the sums of the remaining angles will be equal^c.*

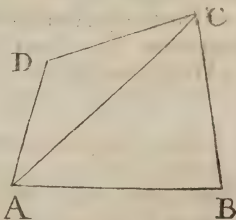
3. *If one angle of a triangle be right, the other two taken together, will be equal to a right-angle.*

4. *The two least angles, of every triangle, are acute.*

THEOREM XI.

The four inward angles of a quadrangle (ABCD) taken together, are equal to four right-angles.

Let the diagonal AC be drawn; then the three angles of the triangle ABC being = two right-angles^d, and those of the triangle ACD equal also to two right-angles^d; it follows that the sum of all the angles of both triangles, which make the four angles of the quadrangle, must be equal to four right-angles^e.



* Ax. 4.

COROLLARY I.

Hence, if three of the angles be right ones, the fourth will also be a right-angle.

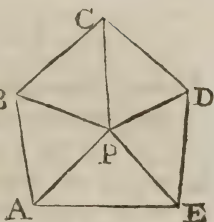
COROLLARY II.

Moreover, if two of the four angles, be equal to two right-angles, the remaining two together will likewise be equal to two right-angles.

SCHOLIUM.

If from any point P, within a polygon ABCDE, lines be drawn to all the angles, so as to divide the

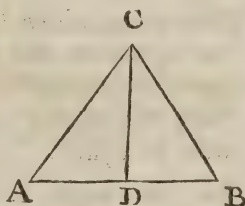
whole into as many triangles APB, BPC, CPD, DPE, EPA, as the polygon has sides; the sum of all the angles of these triangles, (which together make up, or compose the angles of the polygon, over and above those about the point P) will be equal to twice as many right-angles as the polygon has sides (by 10 l.) Therefore, seeing all the angles about the point P, whereby the angles of all the triangles exceed those of the polygon, are equal to four right-angles, it is manifest, that *all the angles of the polygon, taken together, will be equal to twice as many right-angles, wanting four, as the polygon has sides.*



THEOREM XII.

The angles (A, B,) at the base of an isosceles triangle (ABC) are equal to each other.

For, let the line CD bisect, or divide the angle ACB into two equal parts ACD, BCD, and meet AB in D: then the triangles ACD, BCD, having $AC = BC^f$, CD common, and the angle $ACD = BCD^g$, will also have the angle $A = B^h$.



^f Def. 18.

^g Hyp.

^h Ax. 10.

COROLLARY I.

Hence, *the line which bisects the vertical angle of an isosceles triangle, bisects the base, and is also perpendicular to it^h.*

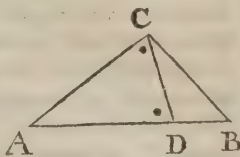
COROLLARY II.

Hence it appears also, that *every equilateral triangle is likewise equiangular.*

THEOREM XIII.

In any triangle ($\triangle ABC$) the greatest side subtends the greatest angle.

Let AB be greater than AC ; in which let there be taken $AD = AC$; and draw CD . The triangle ADC being isosceles, the angles ACD and ADC are therefore equalⁱ; whence ACB , which exceeds the former of them, must also exceed the latter ADC ^k, and consequently, much more exceed B , which is less than ADC ^l.



ⁱ 12. 1.

^k Ax. 2.

^l Cor. to 9. 1.

COROLLARY.

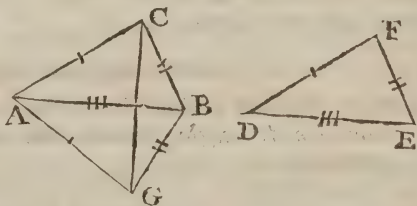
Hence, *in any triangle, the side that subtends the greatest angle, is the greatest*; because ACB cannot be greater than B , unless AB is greater than AC ^m.

^m 13. 1.

THEOREM XIV.

If the three sides (AB, AC, CB) of one triangle, be equal to the three sides (DE, DF, FE) of another triangle, each to each respectively; then the angles opposed to the equal sides will also be equal.

Let the angle $BAG = D$, $AG = DF$, and let GB and GC be drawn; so shall the triangles ABG and DEF be equal in all respects^m:



^m Ax. 10.

ⁿ Hyp.

^o 12. 1.

^p Ax. 4. or 5.

therefore, AG being $= DF = AC$ ⁿ, and $BG = EF = BC$ ⁿ, the angle ACG is also $= AGC$ ^o, and $BCG = BGC$ ^o; and consequently $ACB = AGB$ ^p $= DFE$: therefore the triangles ABC, DEF are equal in all respects^m.

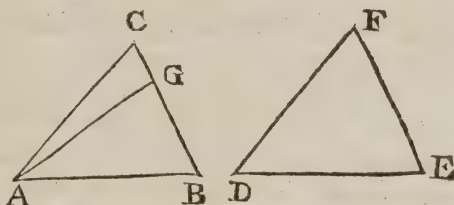
SCHOLIUM.

The demonstration of the last theorem, in obtuse-angled triangles, may admit of another case; which, however, is not necessary; because, if the triangle AGB (equal to DEF) be conceived to be formed on the longest side of ABC; then, all the angles CAB, CBA, GAB, GBA being acute^r, the line CG will, ^r Cor. 4. always fall within the figure ACBG^r, as in the pre- ^{to 10. 1.} sent case. ^r Ax. 2.

THEOREM XV.

If two triangles (ABC, DEF) mutually equiangular, have two corresponding sides (AB, DE) equal to each other, the other corresponding sides will also be equal.

If you say BC is greater than EF; from BC let a part BG be taken = EF^s, and let



^r Post. 4.

AG be drawn. The triangles ABG, DEF having $AB = DE$, $BG = EF$, and $B = E$ (*by hypothesis*), will also have $BAG = D$ ^r; but $D = BAC$ ^u; therefore $BAG = BAC$ ^w; which is impossible.

^r Ax. 10.

^u Hyp.

^w Ax. 1.

and 2.

COROLLARY.

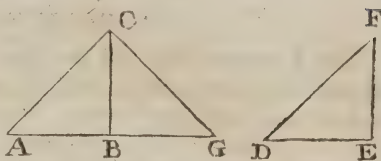
Hence, equiangular triangles, having any two corresponding sides equal, are equal to each other^x. ^x Ax. 10. 1.

THEOREM XVI.

If two right-angled triangles (ABC, DEF) having equal hypotenuses (AC, DF,) have two other sides (BC, EF) likewise equal; the remaining sides (AB, DE) will be equal, and the two triangles equal in all respects.

C

In AB produced,
take $BG = ED$, and
let GC be drawn:
then, the triangles
BCG and DEF,
having $BG = ED$,



^y Hyp.

^z Ax. 7.

^a Ax. 10.

^b 12. 1.

^c Cor. 1.

to 10. 1.

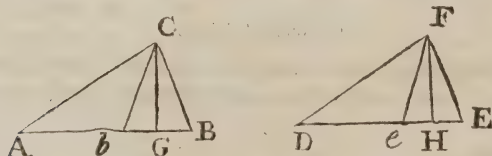
^d 15. 1.

$BC = EF$ ^y, and the angle $CBG = E$ ^z, will also have the angle $G = D$, and $CG = DF$ ^a, $= AC$ ^y: whence, the triangle ACG being isosceles, the angle G , or D , will be $= A$ ^b; and consequently F also $= ACB$ ^c; therefore the triangles ABC and DEF, being mutually equiangular, and having $AC = DF$, they are equal in all respects^d.

THEOREM XVII.

If two triangles (ABC, DEF) having two sides (AC, BC) of the one equal to two sides (DF, EF) of the other respectively, have also the angles (A, D) subtended by two of the equal sides (BC, EF) equal to each other; and if the angles (B, E) subtended by the other equal sides, be either, both acute or both obtuse; then will the two triangles be equal in all respects.

Let CG and FH be perpendicular to AB and



^e Ax. 7.

^f Hyp.

^g 15. 1.

^h 16. 1.

ⁱ Cor. 1. to

10.

DE: then, the angle AGC being $= DHF$, $A = D$, and the side $AC = DF$ ^f, CG will also be $= FH$ ^g; whence, CB being $= FE$ ^f, the angles GBC and HEF are likewise equal^h, and so, the triangles ABC and DEF, being mutually equi-angularⁱ, and having the sides AC and DF equal, are equal in all respects^g.

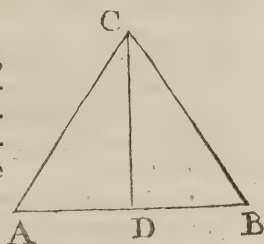
The demonstration is the same, when both the angles are obtuse, as in the triangles $A\hat{b}C$, DeF : for, if $Cb (=CB=FE)=Fe$, the angles $G\hat{b}C$ and HeF being equal (*as before*) the angles $A\hat{b}C$ and DeF will likewise be equal^k.

^k 1. 1. and
Ax. 5. 1.

THEOREM XVIII.

If two angles ($\angle A$, $\angle B$) of a triangle ($\triangle ABC$) be equal, the sides (BC , AC) subtending them will likewise be equal.

Let CD bisect the angle ACB , and meet AB in D : then the triangles ACD , BCD being equiangular^k, and having CD common to both, they will also have $AC=BC$ ^l.



^k Cor. 1. to
10.
^l 15. 1.

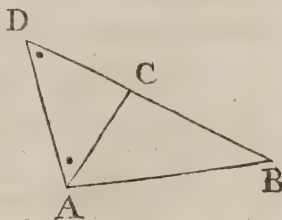
COROLLARY.

Hence, every equiangular triangle, is also equilateral.

THEOREM XIX.

Any two sides (AC , BC) of a triangle ($\triangle ABC$) taken together, are greater than the third side (AB .)

In BC produced, let there be taken $CD=CA$, and let AD be drawn. The angles D and DAC are equal^m; therefore BAD , which exceeds the latterⁿ must also exceed the former D ; and consequently BD (or $BC+AC$) must exceed AB ^o.



^m 12. 1.

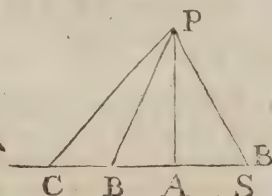
ⁿ Ax. 2.

^o Cor. to
13. 1.

THEOREM XX.

Of all the right-lines (PA, PB, PC) falling from a given point (P) upon an infinite right-line (RS,) that (PA) is the least which is perpendicular to it; and, of the rest, that (PB) which is the nearest the perpendicular is less than any other (PC) at a greater distance.

For BAP being a right-angle^p, ABP will be acute^q, and therefore $AP \sqsubset BP$ ^r.
^p Hyp. to 10. 1.
^q Cor. 4.
^r Cor. to 13. 1.
 Again; when PB and PC are both on the same side of the perpendicular PA; R then is CBP $\sqsubset$ right-angle^s $\sqsubset$ BCP^q, and consequently $PC \sqsubset PB$.

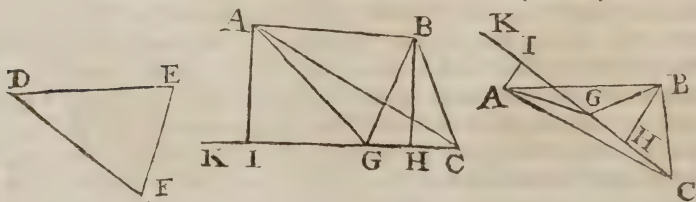


If PB be on the contrary side of the perpendicular to PC; from AC, let AP be taken = AB; then the two lines PB, PB will be also equal^t; and therefore PC, which exceeds the one (by the preceding case) will also exceed the other.
^t Ax. 10.

THEOREM XXI.

Of two triangles (ABC, DEF) having two sides (AB, BC) of the one, equal to two sides (DE, EF) of the other, each to each respectively, the base of that (ABC) will be the greatest which is subtended under the greatest angle,

Let the angle $ABG = E$, $BG = EF$ ($= BC$) also let

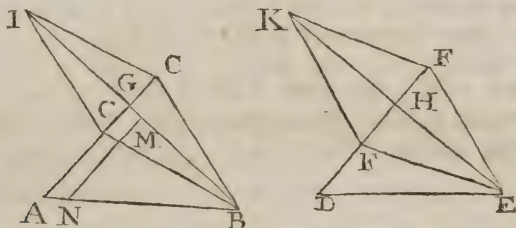


AG and CG be drawn, upon the last of which, produced, let fall the perpendiculars BH and AI^u. ^u4. 1.
 Since BG=BC^x, and, consequently, GH=HC^x, it is ^wHyp.
 evident, that GI (whether the point I be considered as ^x16. 1.
 falling between G and K, or between G and H) will
 be less than CI^z; and therefore AG, or its equal DF^a, ^rAx. 2.
 also less than AC^b. ^aAx. 10.
^b20. 1.

THEOREM XXII.

Of two triangles (ABC, DEF,) having one angle (BAC) in the one equal to one angle (EDF) in the other, and the sides (BC, EF) opposed to them also equal, that (ABC) will have the greatest base, of which the opposite angle (ACB) differs the least from a right-angle.

Let BG and EH be perpendicular to AC and DF, in which produced, take HK=HE, GI=GB, and BM=EH; also let MN be parallel to GA, meeting AB, produced if necessary, in N; and let CI and KF be drawn.

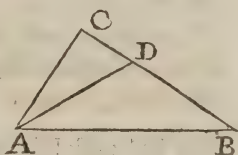


The angle ICG being=BCG^d, and the latter of these greater than EFH^c, (or KFH^d), thence is ICB ^cHyp. $\sqsubset$ KFE; and consequently BI $\sqsubset$ EK; whence, also, ^dAx. 10. BG ($\frac{1}{2}$ BI) $\sqsubset$ EH ($\frac{1}{2}$ EK) or its equal BM^d; and therefore BA $\sqsubset$ BN, because AG and MN being parallels, both the points M and N will fall on the same side of AG. But BN (as the triangles NBM, DEH are equiangular, and have BM=EH^f) is=DE^g: there-^fHyp. and ^g7. 1.
 fore BA is also greater than DE. ^h15. 1.

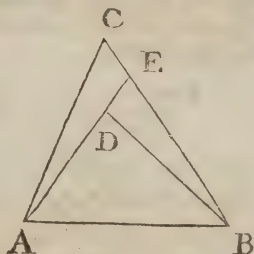
THEOREM XXIII.

If, of two triangles (ABC, ABD) standing upon the same base (AB,) the one be wholly included within the other, the two sides (AD, BD) of the included one taken together, will be less, and the angle (D) contained by them greater, respectively, than the two sides (AC, BC,) and the contained angle (C) of the other.

- CASE I. *If the vertex of the contained triangle be in one side of the other :* Then,
- ^h 19. 1. AD is $\sqsubset$ AC + CD^h; whence, by adding BD common, AD + BD will also be $\sqsubset$ AC + CD + BDⁱ;
- ⁱ Ax. 6. or than its equal AC + BC^k.
- ^k Ax. 3. But the angle ADB is $\sqsubset$ ACB^l.
- ^l Cor. 9. 1.



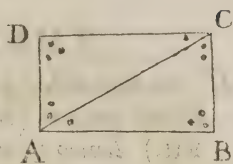
CASE II. *If the vertex be within the other triangle.* Let AD be produced to meet BC in E: then (by case 1.) AD + BD $\sqsubset$ AE + BE, but AE + BE $\sqsubset$ AC + BC, therefore AD + BD $\sqsubset$ AC + BC. Moreover, the angle ADB $\sqsubset$ BED $\sqsubset$ C.



THEOREM XXIV.

The opposite sides (AB, DC) of any parallelogram (ABCD) are equal, as are also the opposite angles (B, D;) and the diagonal (AC) divides the parallelogram into two equal parts.

For AB, DC, and AD, BC being parallelsⁿ, the angle BAC is = DCA^o, and BCA = DAC^e, therefore the equiangular triangles ABC, ADCⁿ, having AC common, are equal in all respects¹.



Def. 24.
7. 1.

Cor. 1. to
10. 1.

15. 1,

COROLLARY.

Hence, if one angle (B) of a parallelogram be a right-angle, all the other three will be right ones: for D, being = B, is a right-angle; and BCD is = B, and DAB = D, by Theor. V.

THEOREM XXV.

Every quadrilateral (ABCD) whose opposite sides are equal, is a parallelogram. (See the preceding scheme.)

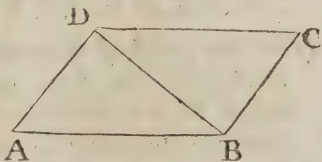
Let the diagonal AC be drawn; then the triangles ABC, ADC being mutually equilateral^r, they will also be mutually equiangular^s; consequently AB will be parallel to DC, and AD to BC^t.

Hyp.
14. 1.
8. 1.

THEOREM XXVI.

The lines (AD, BC) joining the corresponding extremes of two equal, and parallel lines (AB, DC) are themselves equal and parallel.

Let the diagonal BD be drawn. Because AB and DC are parallel^u, the angle ABD is = CDB^w; therefore, BA being = DC^x, and BD common, the remaining sides and angles will likewise be respectively equal^y; and consequently AD parallel to BC^z.



Hyp.
7. 1.
Hyp.

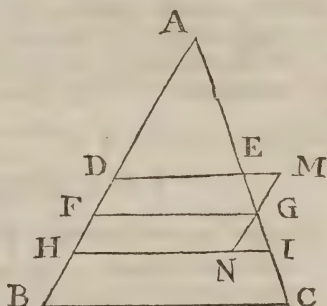
Ax. 10.

8. 1.

THEOREM XXVII.

If, in one side (ΔB) of a triangle (ΔABC), from three points (D, F, H) at equal distances (DF, FH) lines (DE, FG, HI) be drawn parallel to the base, the parts (EG, GI) of the other side (AC) intercepted by them, will also be equal to each other.

Let NGM be parallel to AB , intersecting HI and DE in N and M . Then, the triangles IGN, MGE , having the angle $IGN = EGM^a$, $ING = M^b$, and $GN (= FH^c = FD^d) = GM^c$, will also have $GI = GE^e$.

^a 3. 1.^b 7. 1.^c 24. 1.^d Hyp.^e 15. 1.

COROLLARY I.

Hence it appears, that, *if one side of a triangle be divided into any number of equal parts, and from the points of division, lines be drawn parallel to the base, cutting the other side, they will also divide it into the same number of equal parts.*

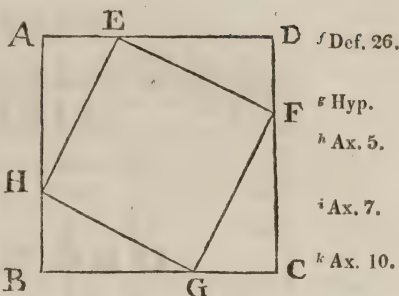
COROLLARY II.

Hence, also, *if two lines FG, HI , cutting the sides of a triangle, be parallel to each other, and another line DE be so drawn as to cut off $FD = FH$ and $GE = GI$, this line DE will be parallel to the two former.*

THEOREM XXVIII.

If in the sides of a square ($\Delta ABCD$), equally distant from the four angular points, there be taken four other points (E, F, G, H), the figure ($EFGH$) formed by joining those points, shall also be a square.

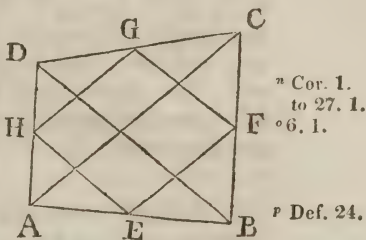
For the wholes AD , DC , CB , BA being equal^f, and also the parts AE , DF , CG , BH ^g, the remaining parts ED , FC , GB , HA must consequently be equal^h; whence, all the angles D , C , B , A being equalⁱ; the sides EF , FG , GH , HE will be equal likewise^k, and $DEF = AHE$ ^k. Therefore, because DEH is $= A + AHE$ ^l, if from these, the equal angles DEF , AHE be taken away, there will remain $HEF = A$ ^h = a right-angle^f. By the same argument (or by Theor. 25th, and the Corol. to the 24th) the other three angles will be right-angles.



THEOREM XXIX.

If all the sides of any quadrilateral ($ABCD$) be bisected, the figure ($EFGH$) formed by joining the points of bisection, will be a parallelogram.

Draw the diagonals AC and BD . Because EF and HG are parallel to AC ⁿ, they are also parallel to each other^o. After the same manner is FG parallel to EH ; therefore $EFGH$ is a parallelogram^p.



THE END OF THE FIRST BOOK.

ELEMENTS

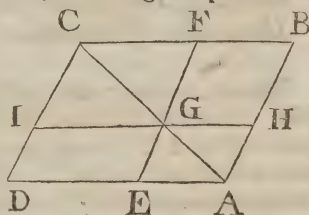
OF

GEOMETRY.

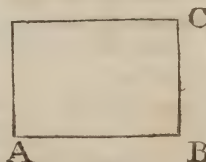
BOOK II.

DEFINITIONS.

1. **I**N a parallelogram ABCD, if two right-lines EF, HI, parallel to the sides, intersecting the diagonal in the same point G, be drawn, dividing the parallelogram into four other parallelograms; those two GD, GB through which the diagonal does not pass, are called *Complements*; and the other two, HE, FI, *parallelograms, about the diagonal*.



2. **E**VERY rectangle is said to be *contained under* the two right-lines AB, BC that form its base and altitude.

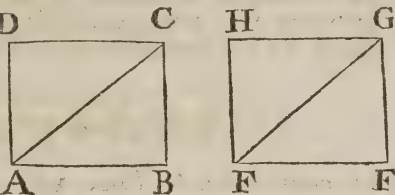


The rectangle contained under two right-lines AB and BC is often, for brevity's sake, denoted by $AB \times BC$. But when the figure is a square, it is usually represented by placing the number 2 over the letter, or letters, expressing the side of the square: thus AB^2 denotes the square made upon the line AB.

THEOREM I.

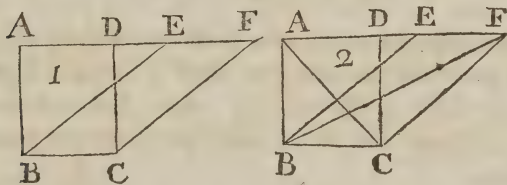
The rectangles (BD, FH) contained under equal lines, are equal.

For let the diagonals AC, EG be drawn: then, because $AB=EF$, $BC=FG$, and $B=F^a$, the triangles ABC, EFG are equal^b. And, in the very same manner will ADC^b Ax. 10. and EHG appear to be equal. Therefore the whole rectangle ABCD is also equal to the whole rectangle EFGH^c.
^a Hyp.
^b Ax. 10.
^c Ax. 4. 1.



THEOREM II.

Parallelograms (ABCD, BCFE) standing upon the same base (BC) and between the same parallel (BC, AF) are equal.



For, since (in Fig. 1.) the angle $F=BEA^d$, and $CDF=A^d$, the triangles FDC, EAB are equiangular^e; they are also equal^f, because $CF=BE^g$: therefore, if each be taken from the whole figure ABCF, there will remain $ABCD=EBCF^h$.
^d Cor. 1. to 7. 1.
^e Cor. 1. to 10. 1.
^f 15. 1.
^g 24. 1.
^h Ax. 5.

COROLLARY I.

Hence, triangles, BAC, BFC, (Fig. 2.) standing upon the same base, and between the same parallels, are also equal, being the halves of their respective parallelogramsⁱ.

ⁱ 24. 1.

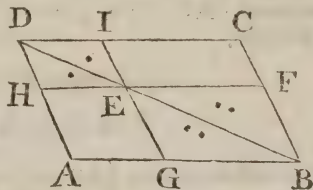
COROLLARY II.

Hence all parallelograms, or triangles, whose bases and altitudes are equal, are equal among themselves; because all such parallelograms are equal to rectangles standing on the same bases, and between the same parallels; and these last are equal, by the preceding proposition.

THEOREM III.

The complements (EC, EA) of any parallelogram (AC) are equal.

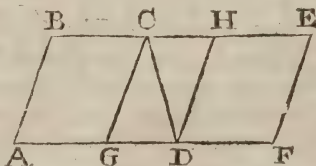
For, the whole triangle BCD being equal to the whole triangle DAB^k, and the parts DIE, EFB respectively equal to the parts DHE, EGB^k, the remaining parts EC, EA, must likewise be equal^l.



THEOREM IV.

A trapezium (ABCD) of which two sides (AD, BC) are parallel, is equal to half a parallelogram, whose base is the sum of those two sides, and its altitude the perpendicular distance between them.

For, in AD produced, take DF = BC; and let CG, DH and FE be all parallel to AB, meeting AF and BC produced, in G, H and E. Then AE is a parallelogram of the same altitude with ABCD, having its base AF equal to the sum of AD and BCⁿ: but this parallelogram, because BG = HF, and CGD = CHD^p, is equally divided by the line CD^q; and so ABCD is the half of it.



ⁿ Constr.

^o Cor 2.
to 2. 2.

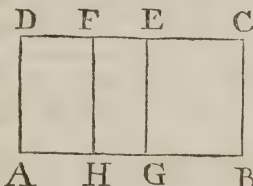
^p 24. 1.

^q Ax. 4.

THEOREM V.

The sum of all the rectangles contained under a given line (AD,) and all the parts (AH, HG, GB) of another (AB,) any how divided, is equal to the rectangle contained under the two whole lines.

Let ABCD be the rectangle contained under the two whole lines, and let HF, GE be parallel to AD, meeting DC in F and E. Then will AF, HE, GC be rectangles ^r of the same altitude with AC ^s; therefore AF = AD × AH, HE = AD × HG, and GC = AD × BG ^t; consequently AD × AB (= AC = AF + HE + GC) = AD × AH + AD × HG + AD × BG ^u.

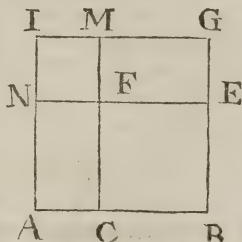


^r Cor. to 24. 1.
^s 24. 1. & Ax. 1.
^t 1. 2.
^u Ax. 3. & 4. 1.

THEOREM VI.

If a right line (AB) be, any-wise, divided into two parts (AC, BC,) the square of the whole line will be equal to the squares of both the parts, together with two rectangles under the same parts.

Let ABGI be the square of AB, and CBEF that of BC, and let EF and CF be produced to meet the sides of the square ABGI in M and N.



From the equal quantities CM, EN ^w take the equal quantities CF and EF, and there remains FM = FN ^x; therefore all the angles of the figure being right ones ^y, NM is a square ^z upon FN (= AC;) and AF, FG are equal to two rectangles under BC and AC ^a: but AG = BF ^a + FI + AF + FG, or AB ² = BC ² + AC ² + 2AC × BC ^b.

^w 24. 1. and Def. fin. 26.
^x Ax. 5. 1.
^y Cor to 24. 1.
^z Def. 26.
^a 1. 2.
^b Ax. 3. 1.

COROLLARY I.

Hence, *the square of any line is equal to four times the square of half that line.*

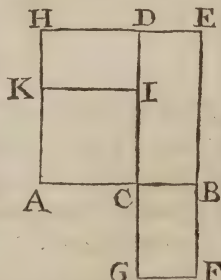
COROLLARY II.

Hence, also, *if two squares be equal, their sides must be equal; because unequal lines BA, BC have not equal squares.*

THEOREM VII.

The difference of the squares (ABEH, ACIK) of any two unequal lines (AB, AC,) is equal to a rectangle under the sum and difference of the same lines.

In EB, produced, take $BF = AC$; let FG be drawn parallel to EH, and let CI be produced both ways, to meet EH and FG in D and G. It is evident that DF is a rectangle^c, whose base GF (= CB^d) = the difference of the given lines AB, AC; and whose altitude FE (because $BE = BA^e$, and $BF = AC^f$) is = the sum of the same lines: but this rectangle DF is = $DB + GB^g = DB + DK$ (because $DK^h = GB$) = square AE -- square AI.



^c Cor. 24.

^d Ax. 24. 1.

^e Def. 24.

^f Hyp.

^g Ax. 3.

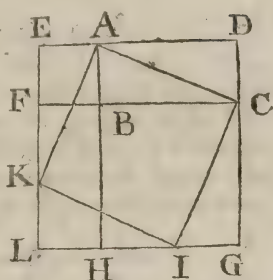
^h 1. 2.

THEOREM VIII.

The square made upon the side (AC) subtending the right-angle of a plane triangle (ABC,) is equal to both the squares (BE, BG) made upon the sides (AC, BC) containing that angle.

Let the sides of the squares BE, BG be produced to meet each other in L and D; in which take KL and IG each equal to AE (or AB;) and let CI, IK, and KA be drawn.

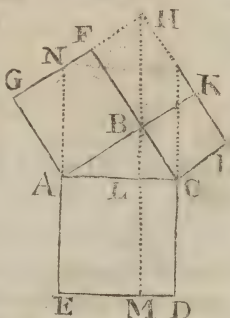
Since ABH and FBC (which are continued right-linesⁱ) are equal to each other^k, EL, DG, ED, and LG will be all equal among themselves^l; and so the angles E, ^{24. 1.} D, G and L being all right ones^m, EDGL will be a ^{5. 1.} square, and consequently ACIK a square likewiseⁿ. ^{28. 1.} Now, if from the square DL, the four equal^o triangles ADC, CGI, ILK, and KEA be taken away, there will remain the square AI: and, if from the same DL, the two equal^p parallelograms DB, BL ^{1. 2.} (which are equal to the said four triangles, because DB=two of them^l) be taken away; then there will remain the two squares BE and BG. Consequently the square AI is=the two squares BE and BG^q. ^{5.}



The same, demonstrated otherwise.

Let AD be the square on the hypotenuse AC, and BG, BI the two squares on the sides AB and BC: let MBH be parallel to AE, meeting GF (produced) in H; and let EA be produced to meet GH in N.

If from the equal^r angles GAB, CAN, the angle NAB, common to both be taken away, there will remain $\angle NAG = \angle BAC$ ^s; whence, as the angle G is also $\angle ABC$ ^r, and the side $AG = AB$ ^t, the sides AN and AC (=AE) are likewise equal^u and therefore the parallelogram AM = the parallelogram AH^w; which last, and consequently the former, is equal to the square BG^x standing on the same base AB, and between the same parallels. By the same argument, the parallelogram CM is=the square BI: and, consequently, the square AD (=AM+CM) = both the squares BG and BI^y.



+

COROLLARY.

Hence, the square upon either of the sides including the right angle, is equal to the difference of the squares of the hypotenuse and the other side^a; or, equal to a rectangle contained under the sum and difference of the hypotenuse and the other side^a.

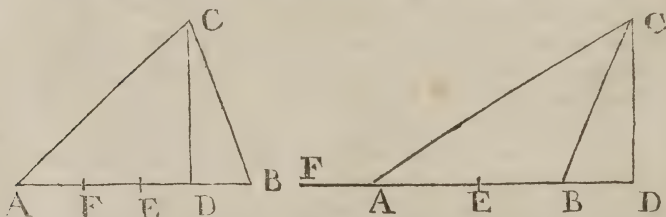
^a Ax. 5.

7. 2.

THEOREM IX.

The difference of the squares of the two sides (AC, BC) of any triangle (ABC) is equal to the difference of the squares of the two lines, or distances (AD, BD) included between the extremes of the base (AB) and the perpendicular (CD) of the triangle,* i. e. $AC^2 - BC^2 = AD^2 - BD^2$.

For, since $AC^2 = DC^2 + AD^2$ and $BC^2 = DC^2 + BD^2$ (by the precedent,) it is evident that the difference of AC^2 and BC^2 will be equal to the difference between $DC^2 + AD^2$ and ^b $DC^2 + BD^2$, or between

^b Ax. 5.

^c Ax. 5. AD^2 and ^c BD^2 , by taking away DC^2 , common, from both.

COROLLARY I.

Since the rectangle under the sum and difference of any two unequal lines, is equal to the difference of their squares^d, it follows, that, the difference of the

^d 7. 2.

* From Pappus, *Math. Coll.* Book VII. Prop. 120.—See also D'Omerique, *Anal. Geom. Introd.* prop. 11., and R. Simson's *Apollonius on Plane Loci*, B. II. Lemma 1.—EDITOR.

squares (or the rectangle under the sum and difference) of the two sides of any triangle, is equal to the rectangle under the sum and difference of the distances included between the perpendicular and the two extremes of the base; that is, $AC^2 - BC^2$
 $(= AC + BC \times AC - BC) = AD + DB \times AD - DB$
 $(= AD^2 - BD^2) = AB \times AE.$

COROLLARY II.

It follows, moreover, that, the difference of the squares (or the rectangle under the sum and difference) of the two sides of a triangle, is equal to twice a rectangle under the whole base, and the distance of the perpendicular from the middle of the base; that is, $AC^2 - BC^2 = AB \times 2DE.$

For, let E be the middle of the base, and make $EF = ED$; then AF being $= BD^c$, the excess of ^{c} Ax. 5. AD above BD (or AF) will (in Fig. 1.) be $= DF = 2DE$; therefore $AD + BD \times AD - BD (= AC^2 - BC^2) = AB \times 2DE.$

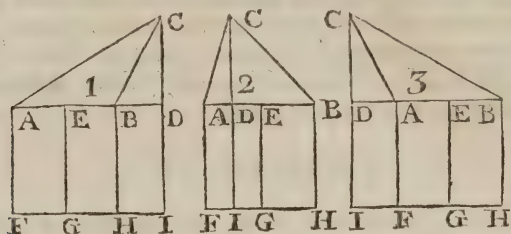
Again (in Fig. 2.) $AD + BD$ being $= AD + AF$ ^{h} Ax. 4. $= FD = 2ED$, and $AD - BD = AB$, we have, also, in this case ^{s} , $AC^2 - BC^2 = AB \times 2DE.$

THEOREM X.

The square of one side (AC) of a triangle (ABC) is greater, or less than the sum of the squares of the base (AB) and of the other side (BC), by a double rectangle under the whole base (AB) and the distance (BD) of the perpendicular from the angle (B) opposite to the side first mentioned; that is, greater, when the perpendicular falls beyond the said angle (as in Fig. 1.); but less, when it falls on the contrary side of that angle (as in Fig. 2, and 3.)

1. 2.

Let the square ABHF, on the base AB, be divided into two equal rectangles EF and EH by



the line EG, bisecting AB in E; and let the perpendicular CD be continued to meet FH (produced) in I.

Cor. 2. In Fig. 1. $AC^2 - BC^2 = 2EI^2 = 2EH + 2BI^m = AH$
 to 9. 2. $(AB^2) + 2BI (2AB \times BD)$; therefore, if from the first
 " Ax. 3. and last of these equal quantities, AB^2 be taken away,
 then AC^2 less both BC^2 and $AB^2 = 2AB \times BD^n$.
 " Ax. 5.

In Fig. 2, and 3. $BC^2 - AC^2 = 2EI^2 = 2BI - 2BG^m$
 $= 2AB \times BD - AB^2$; and so, by adding AB^2 to the
 first and last of these equal quantities, we have here
 " Ax. 4. $AB^2 + BC^2 - AC^2 = 2AB \times BD^o$.

" COROLLARY I.

" Hence, in obtuse-angled triangles (ABC, fig. 1.) the Square (AC) of the side opposite the obtuse-angle (B) is equal to the sum of the squares of the other sides (AB, BC,) together with twice a rectangle contained by one of those sides (AB,) and its continuation (BD) to a perpendicular from the extremity of the other side: that is, $AC^2 = AB^2 + BC^2 + 2AB \times BD$."

" COROLLARY II.

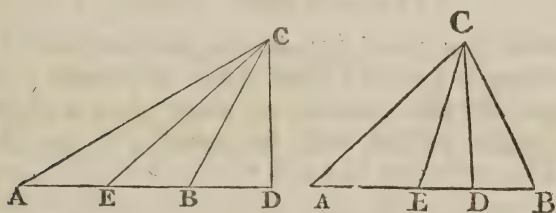
" Hence also, in triangles (figs. 2. and 3.) the squares of the base (AB) and either side, are equal to twice a rectangle contained by the base (AB) and the adjacent segment (BD) made by a perpendicular, together with the square of the other side (AC); that is, $AB^2 + BC^2 = 2AB \times BD + AC^2$."

THEOREM XI.

° 10. 2.

The double of the square of a line (CE) drawn from the vertex to the middle of the base of any triangle (ABC), together with double of the square of the semi-base (AE,) is equal to the squares of both the sides (AC, BC) taken together,* i. e. $2CE^2 + 2AE^2 = AC^2 + BC^2$.

For, let CD be perpendicular to AB: then, because ° AC^2 exceeds the sum of AE^2 and CE^2 (or



BE^2 and CE^2) by the double rectangle $2AE \times ED$ (or $2BE \times ED$); and because BC^2 is less than the same sum by the same double rectangle: it is manifest that both AC^2 and BC^2 together, must be equal to that sum twice taken; the excess on the one part making up the defect on the other.

THEOREM XII.

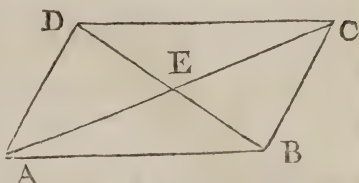
The two diagonals (AEC, BED) of a parallelogram (ABCD) bisect each other; and the sum of their squares is equal to the sum of the squares of all the four sides of the parallelogram.†

* Pappus's *Math. Coll.* VII, 122; R. Simson's *Plane Loci*, Book II, Lem. 6. Schooten's *Exercit. Math.* p. 62.

† Gregory St. Vincent, *de Quad. Circ.*, p. 33.

Garnier, *Reciproques de Geom.* p. 22.

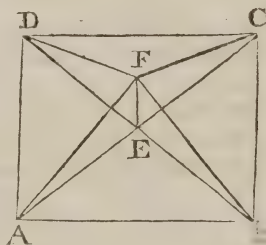
For, the triangles AEB, DEC being equiangular^p, and having $AB=DC$ ^q, will also have $AE=CE$, and $BE=DE$ ^r. Moreover, because $2AE^2 + 2ED^2 = AD^2 + CD^2$, by taking the double of these, we have $4AE^2$ (AC^2) $+ 4ED^2$ (DB^2) $= AD^2 + BC^2 + CD^2 + AB^2$.



THEOREM XIII.

²⁰ 24. 1. and Ax. *If from any point (F), to the four angles of a rectangle (ABCD) four lines be drawn; the sums of the squares of those drawn to the opposite angles will be equal (I say that $FA^2 + FC^2 = FB^2 + FD^2$).*^{*}

For, let the diagonals AC and BD be drawn, bisecting each other in E^x, and let E, F be joined; then the triangles ABC, BAD being equal in all respects^w, thence will AE ($\frac{1}{2}AC$) $= DE$ ($\frac{1}{2}DB$). But $FA^2 + FC^2 = 2AE^2$ ($2DE^2$) $+ 2EF^2 = FB^2 + FD^2$.



^{*} Nov. Act. Acad. Petropol. Tom. 1. 1750.

ELEMENTS

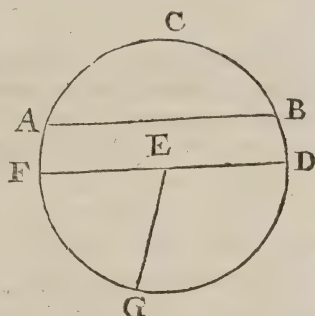
OF

GEOMETRY.

BOOK III.

DEFINITIONS.

1. **ANY** right-line FD , passing through E the center of a circle, and terminating in the circumference at both ends, is called a *Diameter*.



2. An *arch* of a circle, is any portion of the periphery, or circumference, as ACB .

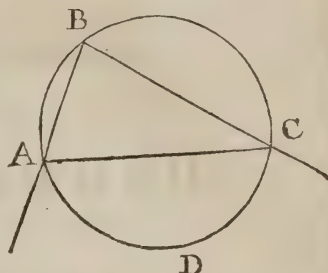
3. The *chord* or *subtense* of an arch ACB , is a right-line AB joining the two extremes of that arch.

4. A *semi-circle* is a figure contained under any diameter and either part of the circumference cut off by that diameter.

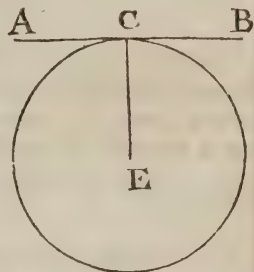
5. A *segment of a circle* is a figure contained under an arch ACB and its chord AB .

6. A *Sector of a circle* is a figure contained under two right-lines EF , EG , drawn from the center to the circumference, and the arch FG included betwixt them. When the two lines EF , EG , stand perpendicular to each other, then the Sector is called a *quadrant*.

7. An *angle* ABC is said to be *in a segment* of a circle ABC, when, being in the periphery of the circle, the right-lines, BA, BC by which it is formed, pass through the extremes of the chord AC bounding that segment.

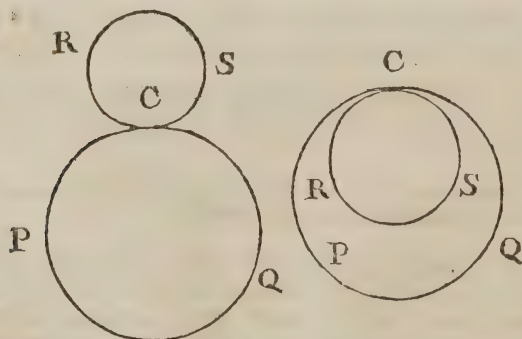


8. An *angle* ABC *in the periphery*, comprehended by two right-lines BA, BC, including an arch of the circle, ADC, is said to *stand upon* that arch.



9. A *right-line* AB is said to *touch* a circle, when, passing through a point (C) in the circumference it cutteth off no part of the circle.

10. Two *circles* (PCQ, RCS) are said to *touch* each other, when the circumferences of both pass



through one point (C) and yet do not cut each other.

11. Two *circles*, in the same plane, are said to *cut* one another, when they fall partly within, and partly without each other; or, when their circumferences cut each other.

12. A *Right-line* is said to be *applied to*, or *inscribed in* a circle, when both its extremes are in the periphery of the circle.

13. A *Right-lined figure* is said to be *inscribed in* a circle, when all its angles are in the circumference of the circle.

14. A *Circle* is said to be *described about* a *right-lined figure*, when the periphery of the circle passes through all the angles of that figure.

15. A *right-lined figure* is said to be *described about* a circle, when all its sides touch the circle.

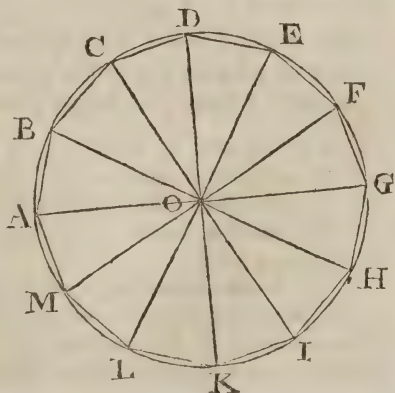
16. A circle is said to be *inscribed in* a *right-lined figure*, when it is touched by all the sides of the right-lined figure.

17. A *right-lined figure* is said to be *inscribed in* a *right-lined figure*, when all the angles of the former are situated in the sides of the latter.

THEOREM I.

If the sides (AB , BC , CD , &c.) of a polygon inscribed in a circle, be equal, the angles ($\angle AOB$, $\angle BOC$, $\angle COD$, &c.) at the center of the circle, subtended by them, will likewise be equal.

For AO , BO , CO , &c. being equal to each other^a, as well as AB , BC , CD , &c. are mutually equilateral; and therefore have all the angles AOB , BOC , &c. equal to each other^b.



^a Def. 33.
of 1.

^b 14. 1.

SCHOLIUM.

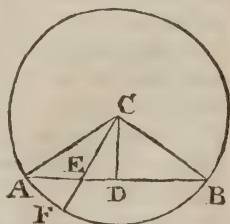
On this proposition depends the division of mathematical instruments for taking and measuring of angles. For if, by repeated trials, or any other means, the circumference of a circle described about a center O , be divided into any number of parts AB , BC , CD , &c. so that the chords be equal; then it is evident, from hence, that all the angles AOB , BOC , COD , &c. which make up the four right-angles AOD , DOG , GOK , KOA at the center, will also be equal to each other, let the radius OA of the instrument be what it will.—In the division of the circle for practical uses, the number of parts into which the circumference is thus divided, or the number of equal angles at the center, is 360; which equal angles are called *degrees*; so that a right-angle, consisting of 90 of these equal angles, is said to be an angle of 90 degrees; every angle being denominated, from the degrees and parts of a degree, which it contains; each degree being conceived to be subdivided into 60 equal parts, called *minutes*; each minute again into 60 equal parts, called *seconds*; and so on to thirds, fourths, fifths, &c. at pleasure.

THEOREM II.

Any chord (AB) of a circle, falls wholly within the same: and a perpendicular (CD) let fall on any chord, from the center of the circle, will divide it into two equal parts.

Let C , A , and C , B be joined; and through any point E in the chord AB , let the right-line CEF be drawn, meeting the circumference in F .

It is evident, because $CA =$
 * Def. 33. 1. CB^c , that these equal lines are on
 different sides of the perpendicular CD^d ; and so, CE being $\sqsupset CA$ or CF^d , the point



E (take it where you will in the line AB) and consequently the line AB itself, will fall within the circle^e. ^e Ax. 2. Moreover, because the triangles ACD, BCD have CA=CB and CD common, thence will AD be also =BD^f. ^f 16. 1.

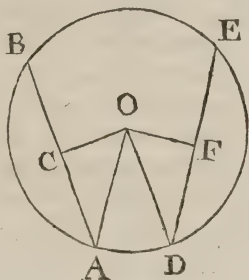
COROLLARY.

Hence a line bisecting any chord at right-angles, passes through the center of the circle.

THEOREM III.

Any two chords (AB, DE) equally distant from the center (O) of a circle, are equal to each other.

Let the perpendiculars OF, OC be drawn, and let O, D and O, A be joined. Because OF=OC^e, OD=OA^f, and F and C are both right-angles^g, therefore is DF=AC^h, and consequently DE=2DFⁱ=2AC^k=ABⁱ.



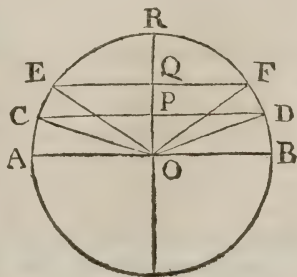
^e Hyp.
^f Def. 33.
of 1.
^g Constr.
^h 16. 1.
ⁱ 2. 3.
^k Ax. 4. 1.

THEOREM IV.

In a circle (AEFB) the greatest line (AB) is the diameter; and of all others terminating in the circumference, that (CD) which is nearest the center (O) is greater than any other (EF) further from it.

1. Draw OC and OD; then it will appear that AB (or OC + OD) > CD^m.

2. Let OP be the distance of CD from the center, and OQ that of EF, both taken in the same radius OR; Draw OE and OF; because the triangles DOC, FOE, have two sides equal



^m 19. 1.

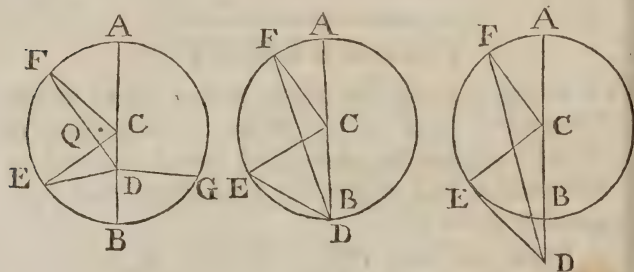
- "Def. 33. 1. each to each", and have the contained angle DOC
 "Ax. 2. $\sphericalangle$ the contained angle FOE^o ; therefore, also, will
 "21. 1. the base DC be $\sphericalangle$ the base FE^p ; and consequently
 $\sphericalangle$ any other chord, at the same distance, with EF^q .
 "3. 3.

COROLLARY.

Hence, a right-line greater than the diameter, drawn from any point within a circle, will cut the circumference.

THEOREM V.

If to the circumference of a circle (AFEB), from any point (D) which is not the center, right-lines (DA , DF , DE) be drawn, the greatest of all (DA) shall be that which passes through the center (C ;) and, of the rest, that (DF) whose other extreme (F) is placed nearest, in the circumference, to the extreme (A) of the greatest, will exceed any other (DE) whose extreme (E) is at a greater distance.



From the center C , let CE and CF be drawn.

- Then, 1. AD ($= \text{DC} + \text{CF}^r$) $\sphericalangle$ DF^s .
 And, 2. Since DC is common, $\text{CF} = \text{CE}$, and
 $\text{DCF} \sphericalangle \text{DCE}^t$, therefore is $\text{DF} \sphericalangle \text{DE}^u$.
 "Ax. 4.
 "19. 1.
 "Ax. 2.
 "21. 1.

COROLLARY I.

Because no two lines, DE , DF , drawn from D , on the same side of the diameter AB , can be equal to

each other^w, three equal right-lines cannot possibly^x 5. 3. be drawn from the periphery to any point, besides the center of the circle: and, therefore, if from a point in any circle, three equal right-lines can be drawn to the periphery, that point is the center of the circle.

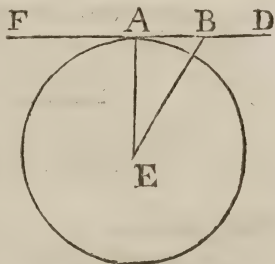
COROLLARY II.

Hence it also follows, that no circle can be described to cut another FBG in more points than two: for, if it were possible to cut it in three points G, E, F, then right-lines drawn from the center Q, to those points, would be all equal^x, which is shown^x Def. 33. 1. to be impossible^y, unless when the center Q coin-^y Cor. 1. to 4. 3. cides with C; and then the circles themselves will neither cut, nor touch, but coincide, and become one circle^x.

THEOREM VI.

A right-line (FD) drawn through any point (A) in the circumference of a circle, at right-angles to the radius (EA) terminating in that point, will touch the circle.

From any point in FD, to the center E, let the right-line BE be drawn; which being greater than AE^a, the point B must, necessarily, fall out of the circle^b; and therefore, as the same argument holds good with regard to every other point in the line FD (except A) it is manifest that this line cuts off no part of the circle, but touches it in one point only.



^a 20. 1.

^b Def. 33.
and Ax. 2.
of 1.

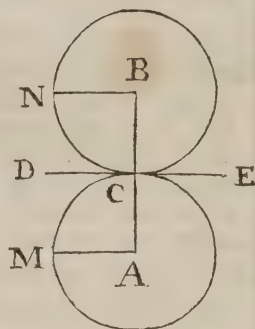
THEOREM VII.

If the distance (AB) of the centers of two circles, be equal to the sum of the two semi-diameters

(AM, BN,) the circles will touch each other, outwardly; and the right-line (AB) joining their centers, will pass through the point of contact.

* Constr.
and
Ax. 5.
d Def. 33.
1.

In AB, take $AC = AM$, and let DCE be drawn perpendicular to AB: then, BC being also $= BN^c$, the circumferences of both circles will pass through the point C^d : but the right-line DE (by the precedent) falls wholly above the one, and wholly below the other; therefore the circles themselves fall wholly without each other, and touch in one point C only.



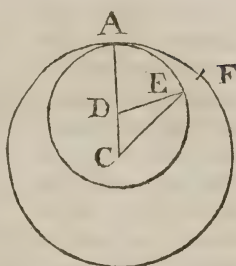
COROLLARY.

Hence, if the centers of two circles be placed at a distance, from one another, less than the sum of the two semi-diameters, a part, at least, of the one will be contained within the other: but, if the distance be greater than that sum, the two circles will then neither touch, nor cut each other.

THEOREM VIII.

If the distance (CD) of the centers of two circles (CAF, DAE) be equal to the difference of the two semi-diameters (CA, DE,) then will those circles touch inwardly; and that radius (CA) of the greater, which is drawn through the center (D) of the less, will meet the two peripheries in the point of contact.

From any point E in the circumference of the less, to the two centers, let EC and ED be drawn. Because CA exceeds DE by the line DC^e, or because $DE + DC = CA^f = DA + DC^g$, therefore is $DA = DE^h$; and so the circumference of the circle D likewise passes through A; but $CA \sqsubset CE^i$: therefore every point in the periphery of the circle D (except A only) falls within the circle C: which was to be demonstrated.



^e Hyp.
^f Ax. 4.
^g Ax. 3.
^h Ax. 5.

ⁱ 5. 3.

COROLLARY I.

Hence, if the centers of two circles be placed at a distance from each other, greater than the difference of the two semi-diameters, a part, at least, of the one will fall without the other; but, if the distance be less than that difference, the lesser circle will then be contained wholly in the greater, but without touching it.

COROLLARY II.

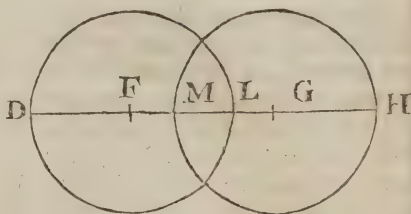
Hence, and from the precedent, it likewise appears, that if two circles touch, either inwardly or outwardly, a right-line, drawn through their two centers, will also pass through the point of contact: because they can only touch, when the distance of their centers is equal to the sum, or to the difference of their semi-diameters^k.

^k Cor. of 7.
 and Cor.
 1. of 8.

THEOREM IX.

If the distance of the centers (F, G) of two circles (DL, MH) be less than the sum, and greater than the difference of the two semi-diameters (FL, GM,) those circles will cut each other.

For, since the distance of the two centers is supposed less than the sum of the semi-diameters, a part of the one circle MH, falls



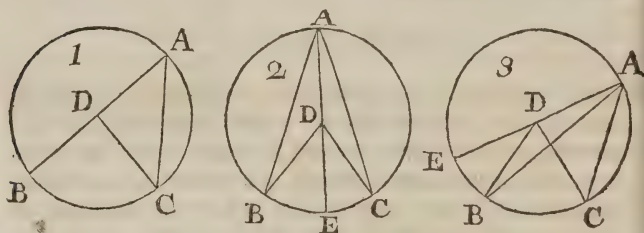
¹ Cor. to 7. within the other DL^1 ; and since that distance is greater than the difference of those semi-diameters, a part of the same circle MH also falls without the

^m Cor. to 8. circle DL^m : which was to be provedⁿ.

³
ⁿ Def. 11.
of 3.

THEOREM X.

The angle (BDC) at the center of a circle, is double to the angle (BAC) at the circumference, when both angles stand upon the same arch (BC).



Let the diameter ADE be drawn.

In the first case (where AB passes through the center) $BDC = A + C^\circ = 2A^\circ$.

^o 9. 1.
^r 12. 1.

In the second case, $BDE = 2BAE$, (by case 1.); to which adding $CDE = 2CAE$, we have $BDC = 2BAC^\circ$.

^q Ax. 4.

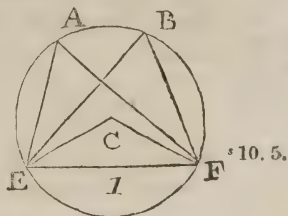
In the third case, $CDE = 2CAE$ (by case 1.) from whence subtracting $BDE = 2BAE$, there remains $BDC = 2BAC^\circ$.

^r Ax. 5.

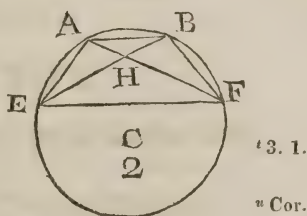
THEOREM XI.

All angles (EAF, EBF) in the same segment, (EABF) of a circle, are equal to each other.

CASE I. *If the segment be greater than a semi-circle; from the center C draw CE and CF; then EAF and EBF being each of them = to $\frac{1}{2}$ ECF^s, they must necessarily be equal to each other.*



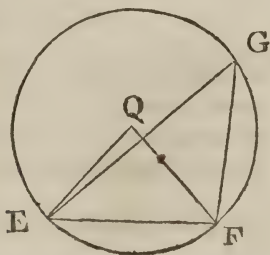
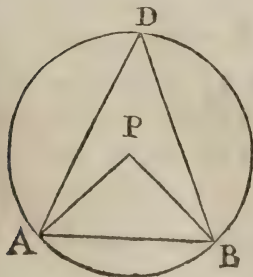
CASE II. *If the segment be less than a semi-circle; let H be the intersection of EB and AF: then the triangles AEH and BFH, having the angle AHE=BHF^t, and AEH=BFH (by case I.) they will also have EAH=FBH^u.*



^u Cor. 1. to 10. 1.

THEOREM XII.

Angles (D, G) in the circumferences, standing upon equal subtenses (AB, EF) of circles having equal diameters, are equal to each other. And the subtenses of equal angles, in the circumferences of circles having equal diameters, are also equal.



From the centers P and Q, let PA, PB, QE, QF be drawn.

^w Hyp.
^x Def. 33
 of 1.
^y 14. 1
^z 10. 3.

1. Hyp. Since $AB = EF^w$, and $AP = BP^x = EQ = FQ^w$; therefore is $P = Q^y$, and consequently $D (= \frac{1}{2} P^z = \frac{1}{2} Q) = G$.

^a Ax. 10.
 of 1.

2. Hyp. Because $D = G$, therefore $P = Q^z$; whence PA being $= QE$, and $PB = QF^w$, AB will also be $= EF^a$.

COROLLARY.

Hence angles in the circumference, standing upon equal chords of the same circle, are equal.

THEOREM XIII.

The angle ($\angle ACB$) in a semi-circle, is a right-angle.

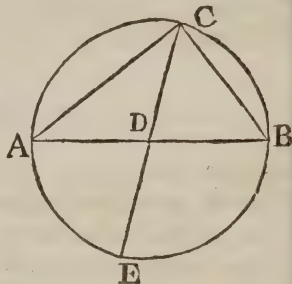
Let the diameter CDE be drawn.

Because $\angle ACD =$ half $\angle ADE$, and $\angle BCD =$ half $\angle BDE^b$, therefore is $\angle ACD + \angle BCD (= \angle ACB) =$ half of $\angle ADE$ and $\angle BDE^c =$ half two right-angles^d $=$ one right-angle.

^b 10. 3.

^c Ax. 4. 1

^d 1. 1.

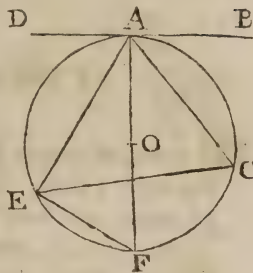


THEOREM XIV.

The angle ($\angle CAB$) included by a tangent to a circle, and a chord (AC) drawn from the point of contact (A), is equal to the angle ($\angle AEC$) in the alternate segment.

Let the diameter AOF be drawn, and E, F be joined

The line DB falling wholly above the circle^e, OA is the least line that can be drawn to it from the center O^f; and OAB is therefore a right-angle^g; but FEA is also a right-angle^h: therefore, if from these equal angles, the equalⁱ angles FAC, FEC (standing on the same arch FC) be taken away, there will remain BAC=AEC^k.



^e Def. 9.
3.

^f Def. 31.
1. and
Ax. 2.

^g 20. 1.

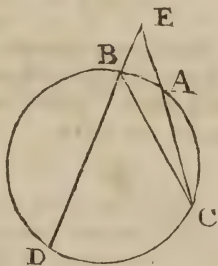
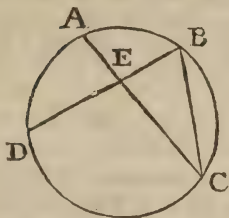
^h 13. 3.

ⁱ 11. 3.

^k Ax. 5.

THEOREM XV.

The angle (DEC) made by two lines (DEB, CEA) intersecting each other within, or without a circle, is, in the former case, equal to the sum, and in the latter, equal to the difference, of two angles in the circumference, standing on the two arcs (DC, AB) intercepted by those lines.



Let the chord CB be drawn.

Then $DEC = DBC + ACB$ ^g, in the first case.

And $DEC = DBC - ACB$ ^h, in the second case.

^g 9. 1.

^h 9. 1. and
Ax. 5.

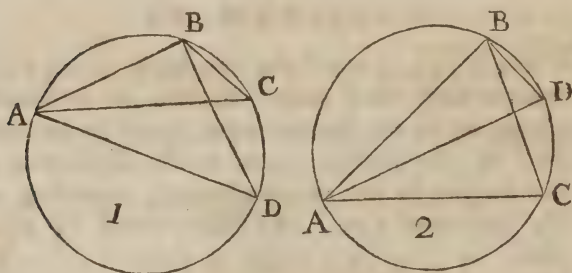
COROLLARY.

Hence, an angle (E) formed below, or above the circumference of a circle, is greater, or less than an angle in the circumference, standing on the same arch.

E

THEOREM XVI.

The vertical angle ($\angle ABC$) of any oblique-angled triangle ($\triangle ABC$) inscribed in a circle ($ABCD$) is greater, or less than a right-angle, by the angle ($\angle CAD$) comprehended under the base (AC) and the diameter (AD) drawn from the extremity of the base.



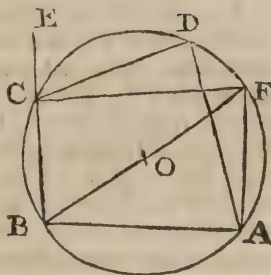
For, BD being drawn, $\triangle ABD$ will be a right-angled,
 and $\angle CAD = \angle CBD^k$; therefore, in the first case, $\angle ABC$
 $= \text{right-angle} + \angle CAD^l$; and in the second, $\angle ABC =$
 $\text{right-angle} - \angle CAD^m$.

THEOREM XVII.

*If any side (BC) of a quadrilateral ($ABCD$) inscribed in a circle, be produced out of the circle, the external angle ($\angle ECD$) will be equal to the opposite, internal angle ($\angle BAD$).**

* From Commandine's Commentary on Prop. 26, B. VII. Pappus's *Mathematical Collections*.

Let the diameter BF be drawn, and let AF and CF be joined: then the angle BAF being a right-angle ($=$ BCF) $=$ ECFⁿ, and DAF also $=$ DCF^o standing both on the same arch DF; thence will the remainders BAD and ECD be also equal^a.



ⁿ 13. 3.

^o 11. 3.

^a Ax. 5.

COROLLARY.

Hence the opposite angles BAD, BCD of any quadrilateral inscribed in a circle, are together, equal to two right-angles. For, since $BAD = ECD$, therefore is $BAD + BCD = ECD + BCD =$ two right-angles^s.

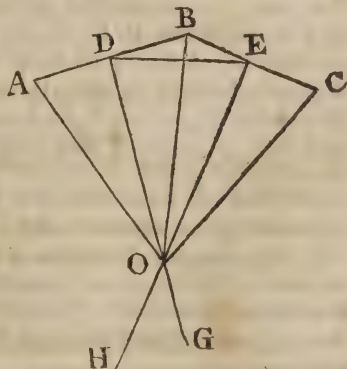
^s 1. 1.

THEOREM XVIII.

Through any three points (A, B, C) not situated in the same right-line, the circumference of a circle may be described.

Draw AB and BC, which let be bisected by the perpendiculars DG and EH, infinitely produced on that side of AB or BC, on which the angle ABC is formed.

These perpendiculars, I say, will intersect each other; and the point of intersection O, will be the center of the circle.



For, if DE be drawn, it is plain, that the angles GDE, HED are less than two right-angles^t; therefore DG, EH, not being parallels^u, they will meet each other^w. Hence, if from the point of intersection O, the right-lines OA, OB, OC be drawn, the triangles ADO, BDO, having two sides equal, each to each^z, and the angles ADO, BDO, contained by them, equal^y, will likewise have $AO = BO$ ^z. After the very same manner is $CO = BO$; therefore $AO = BO = CO$ ^a: whence the circumference of a circle described from the center O, at the distance of AO, will also pass through B and C^b.

^t Ax. 2.^u Cor. 2.

to 7. 1.

^w Def. 13.

of 1.

^z Constr.^y Ax. 7.^z Ax. 10.^a Ax. 1.^b Def. 33.

1.

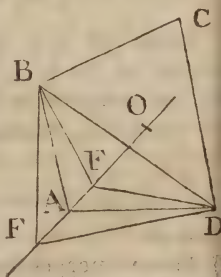
SCHOLIUM.

Hence the method of describing the circumference of a circle through three given points, is manifest.

THEOREM XIX.

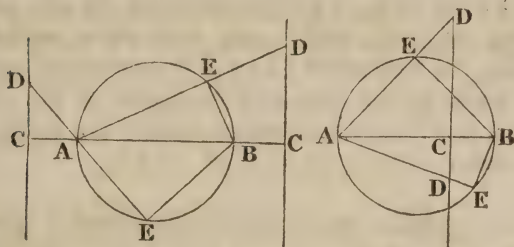
If the opposite angles (BAD, BCD) of a quadrilateral (ABCD) be equal to two right-angles, a circle may be described about that quadrilateral.

For the circumference of a circle may be described through any three points B, C, D, (*by the precedent.*) But, if you deny that it passes through A; then, through the center O, let OAF be drawn, and let it (if possible) pass through some other point F in the line OAF (for it must cut this line somewhere^c;) also let BF' and DF be drawn. Because $BFD + BCD = \text{two right-angles}^d = BAD + BCD^e$; therefore must $BFD = BAD^f$; which is impossible^g. Therefore the circumference of the circle described through B, C and D, must also pass through A.

^c Cor. to 4. 3.^d Cor. to 17. 3.^e Hyp.^f Ax. 5.^g 23. 1.

THEOREM A.

If, through one extremity (A) of a diameter (AB) of a circle (AEB) a right-line (AD) be drawn to meet the circumference (in E) and any perpendicular (CD) to that diameter, (either within or without the circle) the rectangle under the segments (DA, AE) of the line so drawn from the extremity of the diameter, will be equal to the rectangle under the diameter, and the distance of the same extremity from the perpendicular, that is, $DA \times AE = BA \times AC$.*



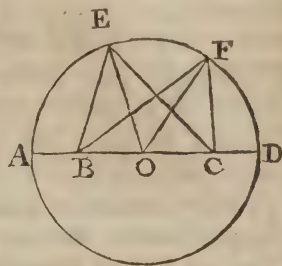
For EB being joined, the angles DCB, DEB are equal, because they are right-angles; therefore a circle may be described through the four points D, C, E, B, (by the last prop.) and consequently the rectangle $DA \times AE$ is equal to the rectangle $BA \times AC$.

THEOREM XX.

If from two points (B, C) in the same diameter (AD) equally distant from the center (O) of a circle, right-lines (BE, CE; BF, CF) be drawn to meet, two by two, in the circumference; the sum of the squares of any two corresponding ones, will be equal to the sum of the squares of any other two, meeting in like manner.

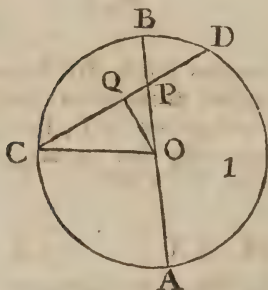
* R. Simson's *Apollonius on Plane Loci*. Prop. 8 and 9.

For, if OE and OF be drawn; then will $BE^2 + CE^2 = {}^h 2BO^2 + 2OE^2$
^{11. 2.} $(2OF^2) = {}^h BF^2 + CF^2$.



THEOREM XXI.

If two lines (AB, CD,) terminated by the periphery on both sides, cut each other within a circle, the rectangle (AP × BP) contained under the parts of the one, will be equal to the rectangle (CP × DP) contained under the parts of the other.



ⁱ 2. 3.

^k Ax. 5.

^l Cor. 1.
to 9. 2.

CASE 1. If one of the two lines (AB) passes through the center O; then let OQ be drawn perpendicular to the other CD, and let OC be joined. It is plain, because $QD = QC$ ⁱ, that DP is equal to the difference of the segments CQ and PQ^k: but the rectangle under the sum and difference of the two sides OC, OP, of any triangle COP, is equal to the rectangle under the whole base CP, and the difference of its two segments^l; therefore, the sum of the two sides OC, OP being $(= OA + OP) = AP$, and their

difference ($=OB-OP$) $=BP$, thence is the rectangle contained under AP and BP equal to the rectangle contained under CP and DP .

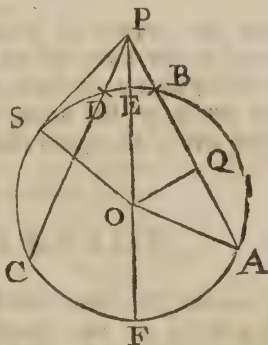
CASE II. If neither of the two lines pass through the center; let the diameter EPF be drawn; then, (by case 1,) $AP \times BP = FP \times EP = CP \times DP$.

THEOREM XXII.

If from two points (A, C) in the circumference of a circle, two lines (AP, CP) be drawn, to pass through and meet without the circle; the rectangle ($AP \times BP$) contained under the whole and the external part of the one, will be equal to the rectangle ($CP \times DP$) contained under the whole and the external part of the other.

Through the center O , let PF be drawn, meeting the circumference in E and F ; let OQ be perpendicular to AP , and let A, O be joined,

Then, (by Cor. 1. to 9. 2.) the rectangle contained under PF ($=PO+OA$) and PE ($=PO-OA$) is = the rectangle contained under AP and PB . After the very same manner $PF \times PE = CP \times DP$: therefore $AP \times BP = CP \times DP$.



^m Ax. 1.

COROLLARY.

Hence, if PS be a tangent at S , and the radius OS be drawn; then, PF being = the sum of PO and OS , and PE = their difference; it follows, that $PS^2 = PF \times PE$ = $PC \times PD$.

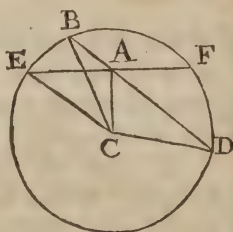
^o Cor. to 8. 2.

THEOREM XXIII.

If from the center (c) of a circle, to a point (A) in any chord (BD), a line (CA) be drawn; the square of that line, together with the rectangle contained under the two parts of the chord, will be equal to a square made upon the radius of the circle.

Let EAF be another chord, perpendicular to CA, and let C, E be joined.

Since $AF = AE$, then will
 $AE^2 = AE \times AF = AB \times AD$;
 to which equal quantities adding AC^2 , we have $CE^2 = AB \times AD + AC^2$.



COROLLARY.

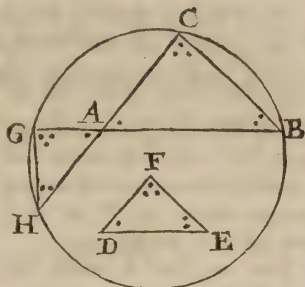
Hence the square of a line (AC) drawn from any point in the base of an isosceles triangle (BCD) to the opposite angle, together with the rectangle of the parts of the base, is equal to a square made upon one of the equal sides of the triangle.

THEOREM XXIV.

The rectangles contained under the corresponding sides of the equiangular triangles (ABC, DEF) taken alternately, are equal.

I say, if $A = D$, $B = E$ and $C = F$, then will $AB \times DF = AC \times DE$.

∴ $AC \times DE = AC \times AH^c = AB \times AG^d = AB \times DF^e$,

^d 21.3.

THEOREM XXV.

^h Cor. 1. to
10. 1.

i 24. 3.

THEOREM XXVI.

The square of a line (CD) bisecting any angle (C) of a triangle ($\triangle ABC$) and terminating in the opposite side (AB), together with the rectangle

($AD \times BD$) under the two segments of that side, is equal to the rectangle of the two sides (CA, CB) including the proposed angle.

Let CD be produced to meet the circumference of a circle, described ^k through the points A, C, B , in E ; and let AE be drawn.

The angles E and B , standing upon the same segment AC , are equal ^l; and ACE is equal DCB (by hypothesis); therefore the triangles AEC, DCB

are equiangular ^m; whereof $AC, CD; CE, CB$ are corresponding sides, opposed to equal angles: therefore $AC \times CB = CD \times CE$ ⁿ $= CD^2 + CD \times DE$ ^o $= CD^2$

+ $AD \times DB$ ^p.

^m Cor. 1.
to 10. 1.

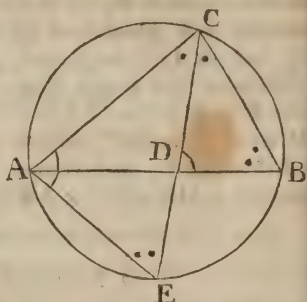
ⁿ 24. 3.

^o 5. 2.

^p 21. 3.

and Ax.

4.



THEOREM XXVII.

The rectangle of the two diagonals (AC, BD) of any quadrilateral ($ABCD$) inscribed in a circle, is equal to the sum of the two rectangles ($AB \times DC, AD \times BC$) contained under the opposite sides.

Let BF be drawn, making the angle $CBF = ABD$, and meeting AC in F .

Because the angle $BCF = ABD$ ^q, and $CBF = ABD$ ^r, the triangles CBF, DBA are equiangular ^s; and therefore $BC, BD; CF, AD$, being corresponding sides, the rectangles $BC \times AD$, and $BD \times CF$ will be equal ^t. Again, the angle ABF being $= CBD$ ^u,

^q 11. 3.

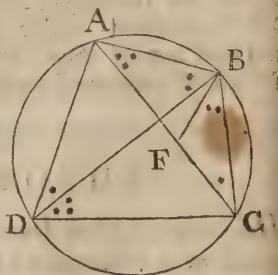
^r Constr.

^s Cor. 1.

to 10. 1.

^t 24. 3.

^u Ax. 4. 1.



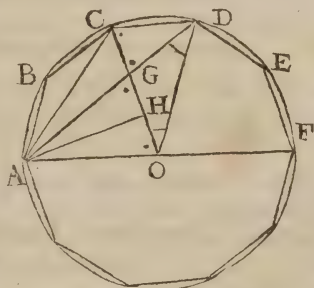
and $\angle BAF = \angle BDC^w$; the triangles ABF and BDC , are, ^w 11. 3. likewise, equiangular; and consequently, AB, BD ; AF, DC being corresponding sides, $AB \times DC = BD \times AF^t$; to which adding $BC \times AD = BD \times CF$ (so proved above,) we have also $AB \times DC + BC \times AD = BD \times AF + BD \times CF = BD \times AC^x$. ^x 5. 2.

THEOREM XXVIII.

If the radius of a circle (OADF) be so divided into two parts, that the rectangle under the whole and the one part shall be equal to the square of the other part; then will this last part be equal to the side (CD) of a regular decagon (ABCDEFG, &c.) inscribed in the circle; and that line whose square is equal to the two squares, of the whole and of the same part, will be equal to the side (AC) of a regular pentagon inscribed in the same circle.

Draw the radii OA, OC, OD, OF ; also draw AD , cutting OC in G , and let AH be perpendicular to OG .

The triangle ODG , having the angle $\angle COD (= \frac{1}{2} \angle DOF^y = \angle OAD^z) = \angle ODA^a$, is isosceles^b: moreover the triangle AOG , having $\angle AOG (= \angle GDO + \angle DOG^c = 2\angle DOC^a) = \angle AOC$, is likewise isosceles^d; as is also the triangle CDG , because, ^d 18. 1. $\angle CGD = \angle AGO^e$, and $\angle CDG (\angle CDA) = \angle FAD^f$, ^e 3. 1. the triangles AOG, CDG are equiangular. Therefore, ^f Cor. to 12. 3. CD, AO ; CG, GO being corresponding sides, we have $CG \times AO (CG \times CO) = CD \times GO^g = GO^2$, be- ^g 24. 3. cause $GO = GD = DC^h$: whence the former part of ^h 18. 1. the proposition is manifest.



^y 1. 3.
^z 10. 3.
^a 12. 1.
^b 18. 1.
^c 9. 1.

† 16. 1.

Again, because $AG=AO$, HG will be $=HO$; and so GC being the difference of the segments HO and HC , we have (*by Cor. 1. to 9. 2.*) $AC^2 - AO^2 = CO \times CG = OG^2$ (*as above*); and consequently $AC^2 = AO^2 + OG^2$.

THE END OF THE THIRD BOOK.

ELEMENTS

OF

GEOMETRY.

BOOK IV.

DEFINITIONS.

A. “ **W**HEN one magnitude contains another a certain number of times, exactly, the former is said to be a *multiple* of the latter, and the latter a *part* of the former.

B. “ When several magnitudes are multiples of as many others, and each contains its part the same number of times, the former magnitudes are said to be *equi-multiples* of the latter, and the latter magnitudes are called *like parts* of the former.

1. *Ratio* is the proportion which one magnitude bears to another magnitude of the same kind, with respect to quantity.

*The measure, or quantity of a ratio, is conceived by considering what part, or parts, the magnitude referred, called the antecedent, is of the other, to which it is referred, called the consequent.**

* 1. Between any two finite magnitudes of the same kind, there subsists a certain relation in respect to quantity, which is called their *Ratio*: The two magnitudes compared together are called the *terms of the ratio*, the first the *antecedent*, and the second the *consequent*.

2. Of four magnitudes, the ratio of the first to the second is said to be *the same as*; or *equal to*, the ratio of the third to the fourth, when the first contains any part, whatever, of the second, just so

A, B, C.
2. 4. 8.

2. *Three quantities*, or magnitudes A, B, C, are said to be *proportional*, when the ratio of the first A to the second B, is the same as the ratio of the second B, to the third C.

A, B, C, D.
2. 4. 5. 10.

3. *Four quantities* A, B, C, D, are said to be *proportional*, when the ratio of the first A to the second B, is the same as the ratio of the third C to the fourth D.

To denote that four quantities A, B, C, D, are *proportional*, they are usually written thus, $A : B :: C : D$; and read thus, as A is to B, so is C to D. But when three quantities A, B, C are *proportional*, the middle one is repeated, and they are written thus, $A : B :: B : C$; " Sometimes, however, they are " written thus, $A : B = C : D$, and then they are read thus, the ratio of " A to B is equal to the ratio of C to D."

4. Of *three proportional quantities*, the *middle one* is said to be a *mean-proportional* between the other two; and the last, a *third-proportional* to the first and second.

5. Of *four proportional quantities*, the *last* is said to be a *fourth-proportional* to the other three taken in order.

A, B, C,
D, E.
1. 2. 4. 8.
16.

6. Quantities are said to be *continually proportional* (or in *continual proportion*) when the first is to the second, as the second to the third, as the third to the fourth, as the fourth to the fifth, and so on; (or, which is just the same thing, when the ratios of the first to the second, of the second to the third, of the third to the fourth, &c., are all equal.)

often as the third contains the like part of the fourth.—But the ratio of the first to the second is said to be *greater* than the ratio of the third to the fourth, when the first contains a part of the second *oftener* than the third contains a like part of the fourth, and *less* when *not so often*.

8. When two ratios are equal, their terms are said to be *proportional*.

7. In a series, or rank of quantities continually proportional, the ratio of the first and third, is said to be *duplicate* to that of the first and second; and the ratio of the first and fourth, *triplicate* to that of the first and second; “also the ratio of the first to the second, is called the *sub-duplicate* of the first to the third. The *sub-triplicate* of the first to the fourth, &c.

8. Any number of quantities, A, B, C, D being given, or propounded, the ratio of the first (A) to the last (D) is said to be *compounded* of the ratios of the first to the second, of the second to the third, and so on to the last.

9. *Ratio of equality*, is that which equal quantities bear to each other.

It may be observed here, that ratio of equality, and equality of ratios, are, by no means synonymous terms: since two or more ratios may be equal, though the quantities compared are all unequal. Thus the ratio of 2 to 1, is equal to the ratio of 6 to 3 (2 being the double of 1, and 6 the double of 3); yet none of the four numbers are equal.

10. *Inverse ratio* is, when the antecedent, is made the consequent, and the consequent the antecedent. *Thus if $2 : 1 :: 6 : 3$; then, inversely, $1 : 2 :: 3 : 6$.*

11. *Alternate ratio* is, when antecedent is compared with antecedent, and consequent with consequent,

As if $2 : 1 :: 6 : 3$; then, by alternation (or permutation) it will be $2 : 6 :: 1 : 3$.

12. *Compound ratio* is, when the antecedent and consequent, taken as one quantity, are compared either with the consequent, or with the antecedent.

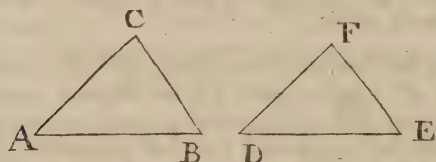
Thus, if $2 : 1 :: 6 : 3$; then, by composition, $2+1 : 1 :: 6+3 : 3$, and $2+1 : 2 :: 6+3 : 6$.

13. *Divided ratio* is, when the difference of the antecedent and consequent is compared, either with the consequent, or with the antecedent.

Thus, if $3 : 1 :: 12 : 4$; then, by division, $3 - 1 : 1 :: 12 - 4 : 4$, and $3 - 1 : 3 :: 12 - 4 : 12$.

These four last definitions, which explain the names given by Geometers to the different ways of managing and diversifying of proportions, are put down here for the sake of order: but are not to be used, or referred to, in any shape, till those properties and relations are demonstrated; which is effected in the three first Theorems of this book.

14. *Similar (or like) right lined figures* are such as have all their angles equal, one to another respectively, and also the sides about the equal angle proportional.



Thus if the angle $A = D$, $B = E$, $C = F$; also $AC : AB :: DF : DE$, $BA : BC :: ED : EF$ &c., then the figures ABC , DEF are said to be similar.

AXIOMS.

1. The same quantity being compared with ever so many equal quantities, successively, will have the same ratio to them all.
2. Equal quantities, have to one and the same quantity, the same ratio.
3. Quantities, having the same ratio to one and the same quantity, or to equal quantities, are equal.
4. Quantities, to which one and the same quantity has the same ratio, are equal.
5. If two quantities be referred to a third, that which is the greatest will have the greatest ratio.

6. If two quantities be *each* referred to a third, that is the greatest which has the greatest ratio.

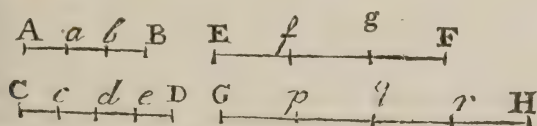
7. Ratios, equal to one and the same ratio, are also equal, one to the other.

8. If two quantities be divided into, or composed of parts, that are equal among themselves, or all of the same magnitude; then will the whole of the one, have the same ratio to the whole of the other, as the number of the parts in the one has to the number of equal parts in the other.

9. If the double, treble, or quadruple, &c. of every part of any quantity be taken, the aggregate will be the double, treble, or quadruple, &c. of the whole quantity propounded.

THEOREM I.

Equimultiples of any two quantities (AB, CD) are in the same ratio as the quantities themselves.



Let the ratio of AB to CD be that of any one number M (3) to any other number N (4), or, which is the same, let AB contain M (3) such equal parts (Aa, ab, bB^a), whereof CD contains the number N (4). Let there be taken Ef, fg, gF any equimultiples of Aa, ab, bB , respectively; and let Gp, pq, qH be the same multiples of Cc, cd, de, eD : shall the whole EF be the same assigned multiple of the whole AB, and the whole GH of the whole CD, as each part in the one, is of its correspondent in the other^b. And, since the parts $Aa, ab, bB, Cc, cd, &c.$ are all equal^c, their equimultiples $Ef, fg, gF, Gp, pq, &c.$ will also be equal^d. Therefore EF is to GH, as the number of the parts in EF

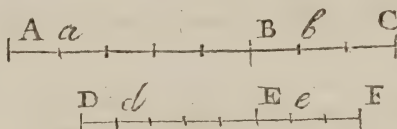
is to the number of equal parts in GH^a , or (which is the same) as the number of parts in AB , to the number of parts in CD , that is, as AB to CD , which was to be demonstrated.

COROLLARY.

Hence, like parts of quantities, have the same ratio as the wholes; because the wholes are equimultiples of the like parts.

THEOREM II.

The two antecedents (AB , DE) of four proportional quantities of the same kind (AB , BC , DE , EF) are in the same ratio with the two consequents (BC , EF .) See Def. 11.



• Ax. 8.
of 4.

1. 4.

Let the common ratio of AB to BC , and of DE to EF , be that of any one number M (5) to any other number N (3); then will AB contain M (5) such equal parts (Aa) whereof BC contains N (3) and DE will, in like manner, contain M (5) such equal parts (Dd), whereof EF contains N (3.) And so AB and DE , as well as BC and EF , being equimultiples of Aa , and Dd , thence will $AB : DE :: Aa : Dd$ (Bb) : Dd (Ee) :: $BC : EF$.

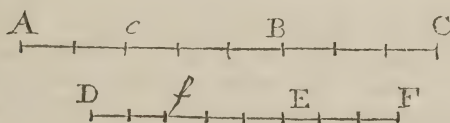
COROLLARY.

The proportionality will subsist, when the consequents are taken as antecedents, and the antecedents as consequents. For $BC : AB ::$ number of parts in BC (or EF) : number of parts in AB (or DE) :: $EF : DE$. See Def. 10.

THEOREM III.

Of four proportional quantities (AB, BC, DE, EF) the sum, or difference, of the first antecedent and consequent ($AB \pm BC$), is to the first antecedent, or consequent, as the sum, or difference, of the second antecedent and consequent ($DE \pm EF$) is to the second antecedent, or consequent. See Def. 12 and 13.

Let what was premised in the demonstration of the preceding theorem, be retained here: then will



C ($AB+BC$) be in proportion to AB , as the number of parts in AC is to the number of equal parts in AB^g , or as the number of parts in DF^g ^{g} Ax. 8. 4. ($DE+EF$) to the number of equal parts in DE , that is, as DF ($DE+EF$) is to DE^g . Again, if from AB and DE , be taken away $Bc=BC$, and $Ef=EF$, then will the difference Ac be in proportion to AB , as the number of parts in Ac (or Df) is to the number of parts in AB (or DE^g ;) that is, as Df is to DE . In the same manner it will appear, that $AB+BC : BC :: DE+EF : EF$; and $AB-BC : C :: DE-EF : EF$.

COROLLARY.

It will appear from hence, that the sum of the greatest and least ($AB+EF$) of four proportional quantities (of the same kind) will exceed the sum ($BC+DE$) of the two means: because, AB being supposed $\sqsubset DE$, Ac will be $\sqsubset Df$, in the same proportion ^{h} : ^{h} 3. 4. and, if to these there be added $BC+EF$ (common;) then will the sum $Ac+BC+EF$ ($AB+EF$) be also $\sqsubset$ the sum $Df+BC+EF$ ($DE+BC$). ^{i} Ax. 6. 1.

SCHOLIUM.

In the demonstration of this and the preceding theorems, the antecedent and consequent are supposed to be divided into parts, all mutually equal among themselves. But it is known to Mathematicians, that there are certain quantities or magnitudes that cannot possibly be divided in that manner, by means of a *common measure*. The theorems themselves are, nevertheless, true, when applied to these *incommensurables*: since no two quantities of the same kind, can possibly be assigned whose ratio cannot be expressed by that of two numbers, so near, that the difference shall be less than the least thing that can be named. But if the matter viewed in this light should not appear sufficiently *scientific*, and you will not (in the preceding theorem) allow the ratio of AC to BC, to be *exactly* the same with that of DF to EF, when AB, BC, and DF, EF, are incommensurables; then let it if possible, be as some quantity aC (less than AC is to BC, so is DE to EF. Let Bb be a part of measure of BC less than the difference (Aa) between AB and aB ; Let Bp be that multiple of B which least exceeds Ba , and let qE be to EF, as p to BC.

It is evident, that pB is $\supset AB$ (because $ap \supset Bb \supset^k Aa$); and that qE is also $\supset DE$, because the ratio of qE to EF, being = that of pB to BC^k, it must necessarily be $\supset$ that of AB to BC^k, or DE to EF, and so $qE \supset DE$ ^m.

Now if (as is supposed) the ratio of aC to BC be = the ratio of DF to EF, it must, of consequence be $\supset$ the ratio of qF to EFⁿ, or (which is the same) than the ratio of pC to BC^k which is impossible.

In like manner it will appear, that no quantity, that is $\supset AC$, can possibly be to BC, as DF is to EF. Therefore $AC : BC :: DF : EF$. And by the same kind of argumentation (authorized and adopted by *Euclid* himself, in his twelfth book) any di-

^k Hyp.

^k Ax. 5. 4.

^m Ax. 6. 4.

ⁿ Ax. 5.

and 7. 4.

^o Proof of
3. 4.

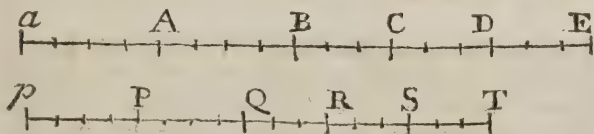
difficulties, or scruples that may be elsewhere brought, from the incommensurability of quantities, may be obviated and removed.

THEOREM IV.

If, of four proportionals (AB, BC, PQ, QR) equimultiples of the antecedents (AB, BC) be taken and compared with any equimultiples of their respective consequents (BC, QR), the ratios will be the same, and the four quantities proportionals.

Let the common ratio of AB, to BC, and of PQ to QR, be that of any one number M to any other number N ; so shall AB contain M such equal parts whereof BC contains N , and PQ, in like manner, M such equal parts whereof QR contains N . Ax. 8. 4.

Let CD, DE be taken each = BC, and RS, ST each = QR, so that BE and QT may be equimultiples of BC and QR; and let CD, DE; RS, ST be conceived to be divided, each into the same number of

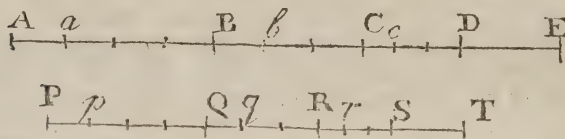


parts with BC, or QR. In like manner, let aB and pQ be taken as equimultiples of AB and PQ, &c. Then will the number of parts in BD = number of parts in QS , and the number of parts in BE = number of parts in QT : and so likewise with respect to aB and pQ . Ax. 4. 1. Therefore aB is to BE , as the number of parts in aB to the number of (equal) parts in BE , or which is the same thing, as the number of parts in pQ to the number of parts in QT , that is, as pQ is to QT , which was to be demonstrated.

THEOREM V.

If of two ranks of quantities (AB, BC, CD ; PQ, QR, RS,) the ratio of the first and second, in the one, be equal to the ratio of the first and second in the other, and the ratio of the second and third, in the one, equal likewise to the ratio of the second and third in the other ; then, also, shall the ratio of the first to the third, be the same in the one rank as in the other.

Let the common ratio of AB to BC, and of PQ to QR, be still expressed as in the preceding demonstrations. Let, moreover, CD and RS be conceived to be divided, each into the same number of parts with BC and QR.



Because the quantities BC, CD, QR, RS, are proportional^r, their like parts Bb, Cc, Qq, Rr (being in the same ratio with the wholes^s) will also be proportionals ; or, because $Aa = Bb$ and $Pp = Qq$ ^r, it will be^t $Aa : Cc :: Pp : Rr$. But AB and PQ are *equimultiples* of the antecedents Aa and Pp^r ; and CD, RS are *equimultiples* of the consequents Cc and Rr ; therefore^u $AB : CD :: PQ : RS$; * *which was to be demonstrated.*

^r Hyp.

^s Cor. to

1. 4.

^t Ax. 2. 4.

^u 4. 4.

* When in two ranks of quantities, the proportions are inordinate, as $AB : BC :: QR : RS$, and $BC : CD :: PQ : QR$; the same thing may be demonstrated ; and that in the very same manner, except only that QR must here be divided into the same number of parts with AB, and PQ.

COROLLARY I.

If other quantities DE, ST be taken, still proportional to the two next preceding them, so that $CD : DE :: RS : ST$; then, by the same argument (regard being had to AB, CD, DE in the one rank, and PQ, RS, ST in the other) it is evident that $AB : DE :: PQ : ST^x$. And thus we may go on, ^x 5. 4. still assuming other quantities, as many as we please; and the ratio of the first and last, will always be the same in one rank, as in the other. Therefore *ratios^y compounded of the same number of like, or^y equal ratios, are equal.* Def. 8. of 4.

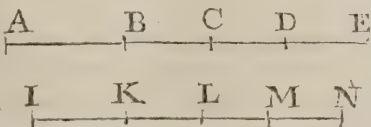
COROLLARY II.

It is also evident from hence, that *if any two quantities be taken, proportional to the two consequents of an assigned proportion, they will also be proportionals when compared with the antecedents; and vice versâ.* For, the two quantities CD and RS, when compared, successively, with the consequents and antecedents of the given proportion $AB : BC :: PQ : QR$, appear to be proportional, in the one case, as well as in the other^z. ^z 5. 4.

THEOREM VI.

If, to the two consequents (BC, KL) of four proportionals (AB, BC, IK, KL,) any two quantities (CD, LM) that have the same ratio to the respective antecedents be added; these sums and the antecedents will still be proportionals (I say, if $AB : BC :: IK : KL$, and $AB : CD :: IK : LM$; then shall $AB : BD :: IK : KM$.)

For CD and LM being proportional to the antecedents AB and IK^a, they and the consequents (BC,



^a Hyp.

^b 3. 4. KL) will also be proportionals (*Corol. 2 of the last*): whence^b (*by comp.*) $BC : BD :: KL : KM$. And so again (*same Corol.*) $AB : BD :: IK : KM$.

COROLLARY.

From this theorem it will appear, that, *if the ratios of the corresponding quantities of two ranks with respect to the two first, are the same in both ranks* ($AB : BC :: IK : KL$, $AB : CD :: IK : LM$, &c.;) *then the ratio of all the quantities to the first, will also be the same in the one rank, as in the other.* For, by adding DE and MN to the last consequents (BD, KM) there will be had^c $AB : BE :: IK : KN$ (and so on, to any number of quantities whatever.) Then (*by comp.*) $AB : AE :: IK : IN$.

^c 6. 4.

When the quantities in both ranks, are of the same kind, it will appear (*by alternation*) that the ratio of the two sums, and *that of every two corresponding terms*, will be the same.

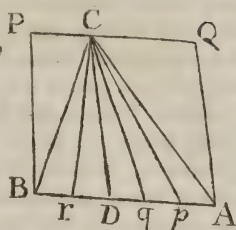
The six theorems here delivered, on proportions of magnitudes in general, comprehend all that is most useful in that subject. — What relates to the proportions of extended magnitudes, under different limitations, and figures, as far as regards right-lines and surfaces, will be the subject of the remaining part of this book.

Note, whenever you meet with several ratios connected by the sign :: (like these, $A : B :: C : D :: E : F :: G : H$) the conclusion to be drawn, is always from the first and last of the two equal ratios.

THEOREM VII.

Triangles (ACD, BCD,) and also parallelograms (ADCQ, BDCP,) having the same altitude, are to one another in the same ratio as their bases (AD, BD).

Let the base AD be to the base BD in the ratio of any one number m (3) to any other number n (2,) or which is the same^c, let AD contain m (3) such equal parts whereof BD contains the number n (2.) Then, the triangles ACp, pCq, BCr, &c. made by drawing lines from the points of division to the vertex C, being all equal^d; the tri-



^c Ax. 8. 4.

angle ACD will be to the triangle BCD, as the number of equal parts in the former, to the number of equal parts in the latter, or as the number of parts in AD to the number of parts in BD, that is, as AD to BD^e; whence, also, the parallelograms ADCQ, DCP, being the doubles of their respective triangles^d, are likewise in the same ratio as their bases AD and BD^e.

^d Cor 2. to 2. 2.

^e 4. 4.

SCHOLIUM.

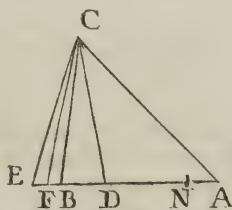
If the bases AD and BD are incommensurable to each other, the ratio of the triangles cannot be other than that of their bases.

For, if possible, let the triangle BCD be to the triangle ACD, not as BD to AD, but as some other line ED, greater than BD, to AD.

Let AN be a part, or measure

of AD, $\supset$ BE^e, and let DF be that multiple of AN which least exceeds DB; also let CE and CF be drawn. It is

manifest, that the point F falls between B and E, because (by Hyp.) BF $\supset$ AN, and AN $\supset$ BE. Moreover, the ratio of FCD to ACD is the same as that of ED to AD (by the precedent.) But the ratio of BCD to ACD (or of ED to AD^f) is $\supset$ the ratio of FD to AD^g, or of FCD to ACD; and consequently BCD $\supset$ FCD^h; which is impossibleⁱ; by the same argument it will appear, that the triangle BCD cannot be to the triangle ACD, as a line, less than BD is to AD. Therefore BCD : ACD :: BD : AD.



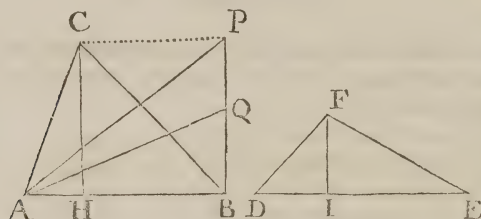
^f Lem. 1. 8.

^g Hyp. Ax. 5. of 4. Ax. 6. of 4. Ax. 2. of 1.

If this Scholium should appear difficult to the learner, it may not be amiss to omit it entirely; since it is only put down for the sake of those who may be scrupulous about the business of incommensurables; to whom it may not be improper to observe, that nothing more is taken for granted here, than what is effected by means of the first Lemma in the 8th book; which, being demonstrated from axioms, and one single theorem in the first book, is referred to, here, though not given till hereafter, for reasons already hinted at in this note.

THEOREM VIII.

Triangles (ABC, DEF) standing upon equal bases (AB, DE) are to one another, in the same ratio as their altitudes (CH, FI.)



Let BP be perpendicular to AB, and equal to CH; in which let there be taken BQ=FI, and let AP and AQ be drawn.

^k Cor. 2.
to 2. 2.
17. 4.

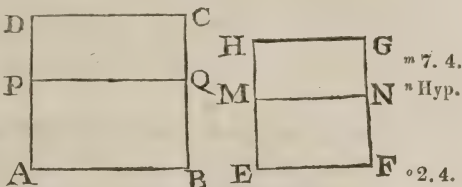
The triangle $ABP = ABC^k$, and $ABQ = DEF^k$, but $ABP (ABC) : ABQ (DEF) :: BP (HC) : BQ (FI)$

THEOREM IX.

If, parallel to the bases of any two parallelograms (AC, EG,) two lines (PQ, MN) be drawn, so as to cut the sides proportionally (AP : AD

$:: EM : EH$,) then will those parallelograms and their corresponding parts (AQ, EN) be also proportionals.

For, $AQ : AC :: D$
 $AP : AD^m :: EM :$
 $EH^n :: EN : EG^m :$
 and therefore, by
 alternation, $AQ :$
 $EN :: AC : EG^o :$

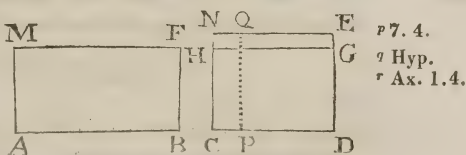


THEOREM X.

If four lines are proportional ($AB : CD :: DE : BF$,) the rectangle (AF) under the two extremes, will be equal to the rectangle (CE) under the two means. And, if the rectangles under the extremes and means of four given lines (AB, CD, DE, BF) be equal, then are those four lines proportional.

In DE let DG be taken = BF, and let GH, parallel to DC, be drawn.

1. Hyp. $AF : CG$
 $: AB : CD^p :: DE :$
 $BF^q :: DE : DG^r ::$
 $CE : CG^p$; therefore,
 the consequents of the
 first and last of these
 equal ratios being the same quantity CG, the two antecedents AF and CE must be equal^s.



2. Hyp. $AB : CD :: AF : CG^p :: CE : CG^t :: DE :$
 DG^p (BF.)

SCHOLIUM.

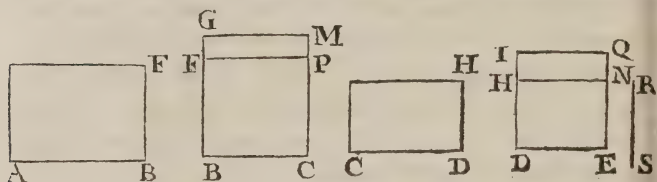
From the same demonstration and scheme, it will appear, that the two antecedents of four proportional lines (AB, CD, DE, BF) are in the same ratio to

each other, as the two consequents: for, if in DC there be taken $DP=BF$, and PQ be drawn parallel to DE :
 ' 7. 4. then $AB : DE :: AF : PE^t :: CE : PE :: CD : PD$
 (PF.)

THEOREM XI.

The rectangles under the corresponding lines, of two ranks of proportionals, are themselves proportionals. (I say, if $AB : BC :: CD : DE$, and $BF : BG :: DH : DI$, then will the rect. $AF : \text{rect. } BM :: \text{rect. } CH : \text{rect. } DQ$.)

For, in BG and DI (produced if necessary) let there be taken $BF=BF$, $DH=DH$, and let FP be parallel



^y 7. 4. to BC , and HN to DE : then $AF : BP :: AB : BC^y ::$
^z Hyp. $CD : DE^z :: CH : DN^z$; whence (*alternately*) $AF :$
^a 9. 4. $CH :: BP : DN$, and so likewise is BM to DQ^a :
 whence (*again by alternation*) $AF : BM :: CH : DQ$.

COROLLARY I.

Hence, the squares of four proportional lines, are themselves proportional.

COROLLARY II.

Hence also, the sides of four proportional squares (AB^2, BC^2, CD^2, DE^2) will be proportional. For let the line RS be taken such, that $AB : BC :: CD : RS$; then, since $AB^2 : BC^2 :: CD^2 : RS^2$ (*by Corol. 1.*) and $AB^2 : BC^2 :: CD^2 : DE^2$ (*by supposition*) thence will ^b $RS^2 = DE^2$; therefore ^c $RS = DE$, and consequently $AB : BC :: CD : DE^d$.

^b Ax. 4. 4.

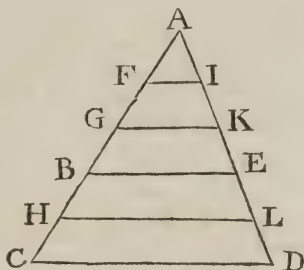
^c Cor. 2.
 of 6. 2.

^d Ax. 1. 4.

THEOREM XII.

A line (BE) drawn parallel to one side (CD) of a triangle (ACD) divides the other two sides proportionally. (I say, $AB : AC :: AE : AD$, $AB : BC :: AE : ED$, and $AC : BC :: AD : ED$.)

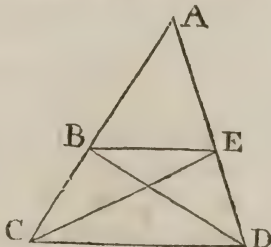
Let AB be to AC, as any one number m (3) is to any other number n (5), or, which is the same, let AB contain m (3) such equal parts whereof AC contains n (5.) Then, if from the points of division, lines be drawn parallel to the side CD, they will also divide AE and AD into the like number of equal parts^e, ^e Cor. 1. therefore AE is to AD, as the number of equal parts to 27. 1. in AE to the number of equal parts in AD, or as the number of equal parts in AB to the number of equal parts of AC, that is, as AB to AC^f. In the same^f Ax. 8. manner, AE is to ED, as the number of parts in AE to the number of parts in ED, or as the number of parts in AB to the number of parts in BC, that is, as AB to BC. Also, in the same manner, $AC : BC :: AD : ED$.



The same otherwise.

Draw CE and DB. Then will the triangles BEC and EBD be equal to each other^f; whence, by add-^f Cor. 1. ing BEA to both, AEC will be also = ABD^g. ^{to 2. 2.} ^{g Ax. 4. 1.}

But, $AB : AC :: \text{triang. AEB} : \text{triangle AEC}$ ^h (ABD) $:: AE : AD$ ^h; and $AB : BC :: \text{triang. AEB} : \text{triangle CEB}$ ^h (DEB) $:: AE : ED$.



^h 7. 4.

COROLLARY.

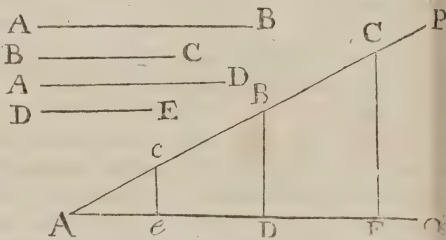
Hence a right line, which divides two sides of a triangle proportionally, is parallel to the remaining side: because AD is divided in the same ratio with AC, when BE is parallel to CD; but not elseⁱ.

ⁱ Ax. 2. 1.
and 5. 4.

SCHOLIUM.

From this last theorem, whatever relates to the composition and division of ratios, when these respect the comparison of right-lines, will appear exceedingly obvious.

For, let AB, BC, AD, and DE, be proportionals: and from any point A, let two indefinite right-lines AP, AQ be drawn; in



which take $AB=AB$, $BC=BC$, $AD=AD$, and $DE=DE$; also take $Bc=BC$, $De=DE$, and let BD , CE , and ce be drawn.

^{*} Hyp.

ⁱ Cor. of
12. 4.

^m Cor. 2.
to 27. 1.

Since $AB : BC :: AD : DE$ ^k, thence is EC parallel to DB ⁱ; and so, Bc being $=BC$, and $De = DE$, ec will also be parallel to DB ^m. Therefore,

$$AC (AB+BC) : AB :: AE (AD+DE) : AD;$$

$$AC (AB+BC) : BC :: AE (AD+DE) : DE;$$

$$Ac (AB-BC) : AB :: Ae (AD-DE) : AD;$$

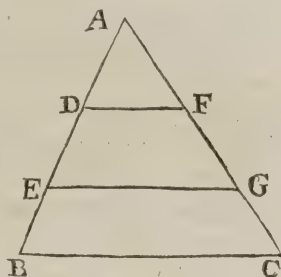
$$Ac (AB-BC) : BC :: Ae (AD-DE) : DE;$$

$$\text{And, } AC (AB+BC) : Ac (AB-BC) :: AE (AD+DE) : Ae (AD-DE).$$

THEOREM XIII.

The parts (DE, FG) of the two sides of a triangle, intercepted by right-lines (DF, EG) drawn parallel to the base (BC), are in the same ratio with the wholes. ($DE : FG :: AB : AC$).

For, DF, and EG being parallel to each other, thence will $DE : AE : FG : AG$ ^o; therefore, by (alternation) $DE : FG :: AE : AG$ ^p. In the same manner, $AE : AG :: AB : AC$ ^o. Consequently $DE : FG : AB : AC$ ^q.



^o 12. 4.

^p 2. 4.

^q Ax. 7. 4.

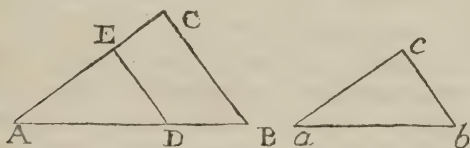
COROLLARY.

Hence, if ever so many lines be drawn parallel to the base, cutting the sides of a triangle, every two corresponding segments will have the same ratio ^q.

THEOREM XIV.

In triangles (ABC, abc) mutually equiangular, the corresponding sides (AB, ab, AC, ac) containing the equal angles (A, a) are proportional.

In AB and AC (produced if necessary) take AD = ab, and AE = ac, and join D, E.



The triangles abc and ADE having $ab = AD$, $ac = AE$, and the angle $a = A$, have also the angle ADE ^r Ax. 10. $\angle abc = \angle ABC$ ^s; whence DE will be parallel to BC ^t; and therefore $AB : AD (ab) :: AC : AE (ac)$. ^u

The same otherwise.

Because $AB \times ac = ab \times AC$ ^w, therefore is $AB : ab$ ^w 24. 3. $: AC : ac$ ^x.

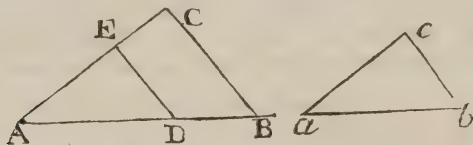
COROLLARY.

Hence equiangular triangles are similar to each other ^y. ^y Def. 14. of 4.

THEOREM XV.

If two triangles ($\triangle ABC$, $\triangle abc$) have one angle ($\angle ABC$) in the one, equal to one angle ($\angle bac$) in the other, and the sides (AB , ab , AC , ac) about those angles proportional; then are the triangles equiangular.

In AB and AC , take $AD=ab$, and $AE=ac$, and let DE be drawn.



^b Hyp.

^c Cor. to

12. 4.

^d Cor. 1. to

7. 1.

^e Ax. 10.

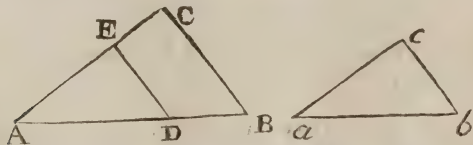
1.

Since $AB : ab$ (AD) $:: AC : ac$ (AE ^b), therefore DE is parallel to BC ^c; whence the angle $B = ADE = b$ ^e, and the angle $C = AED = c$.

THEOREM XVI.

If two triangles ($\triangle ABC$, $\triangle abc$) have one angle ($\angle A$) in the one, equal to one angle ($\angle a$) in the other, and the sides (AB , ab , CB , cb) about either of the other angles proportional; then will the triangles be equiangular, provided these last angles (B , b) be, either, both less, or both greater, than right-angles.

In AB , let AD be taken $=ab$, and let DE be drawn parallel to BC , meeting AC in E .



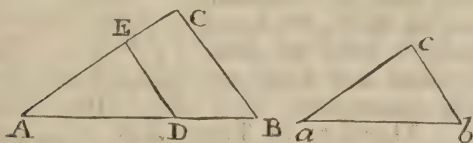
Then will the triangles ABC , and ADE , be

equiangular^f; therefore, $CB : ED :: AB : AD$ ^{g^f Cor. 1.}
 $: AB : ab$ ^h $:: CB : cb$ ⁱ; and consequently $ED =$ ^{to 7. 1.}
 ab ^k; whence the triangles abc and ADE (having $ab =$ ^{14. 4.}
 AD , $cb = ED$, and $a = A$) will be equal in all re- ^{h Ax. 1. 4.}
 spects^l, provided the angles abc and $ABC (= ADE)$ ^{i Hyp.}
 are either both less or both greater than right-angles. ^{k Ax. 4. 4.}
 Therefore, since the latter of these equal triangles ^{l 17. 1.}
 abc , ADE) is equiangular to ABC , the proposition is
 manifest.

THEOREM XVII.

If two triangles (ABC , abc) have all their
 sides, respectively proportional ($AC : ac ::$
 $AB :: ab :: CB : cb$) then are those triangles
 equiangular.

In AC and AB , take $AE = ac$, and $AD = ab$, and
 in E , D .

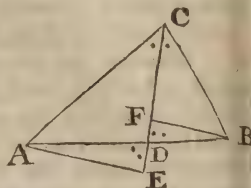


Since $AC : AE (ac) :: AB : AD (ab)$ ^m, the triangles ^{n Hyp.}
 ABC , ADE are equiangularⁿ; hence $CB : ED ::$ ^{15. 4.}
 $AB : AD$ ^o (ab) $:: CB : cb$ ^m; and consequently ^{14. 4.}
 $ED = cb$ ^p; therefore the triangles abc , ADE , being ^{p Ax. 4. 4.}
 mutually equilateral, they must also be mutually equi-
 angular^q; and consequently abc , as well as ADE , ^{q 14. 1.}
 equiangular to ABC .

THEOREM XVIII.

A right-line (CD) bisecting any angle (ACB) of
 a triangle (ABC) divides the opposite side (AB)
 into two segments (AD , BD) having the same
 ratio with the sides (AC , CB) containing that
 angle.

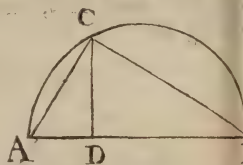
Let AE and BF be perpendicular to CDE. Then the triangles ACE, CBF, and ADE, BDF being, respectively, equiangular^r, it will be $AD : BD :: AE : BF$ ^s
^{14. 4.} and $2. 4.$, $AC : BC$.



THEOREM XIX.

A perpendicular (CD) let fall from the right angle (C) upon the hypotenuse (AB) of right-angled triangle (ABC) will be a mean proportional between the two segments (AD, BD) of the hypotenuse; and each of the sides containing the right-angle, will be a mean proportional between its adjacent segment, and the whole hypotenuse.

For since the angle BDC is $\angle BCA$ ^t, and B common, the triangles BDC, BCA, are equiangular^u: after the same manner ADC and ABC appear to be equiangular.



Therefore, by Theor. XIV.

$$BD : CD :: CD : AD$$

$$AB : BC :: BC : BD$$

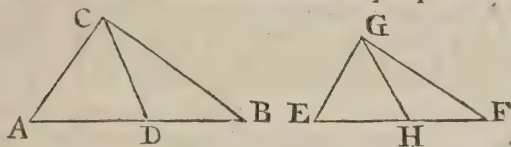
$$AB : AC :: AC : AD$$

COROLLARY.

^{13. 3.} Because the angle in a semi-circle is a right-angle it follows, that, if from any point C, in the periphery of a semi-circle ACB, a perpendicular CD be let fall upon the diameter, AB, and from the same point C, to the extremities of that diameter two chords CA, CB be drawn; the square of the perpendicular will be equal to a rectangle under the two segments of the diameter; and the square of each chord, equal to a rectangle under the whole diameter and its adjacent segment: for because the above proportions, we have $CD^2 = BD \times AD$ ^{10. 4.} $BC^2 = AB \times BD$, and $AC^2 = AB \times AD$.

THEOREM XX.

If, in similar triangles ($\triangle ABC$, $\triangle EFG$) from any two equal angles ($\angle ACB$, $\angle EGF$) to the opposite sides, two right-lines (CD , GH) be drawn making equal angles with the homologous sides (CB , GF); those right-lines will have the same ratio as the sides (AB , EF) on which they fall, and will also divide those sides proportionally.

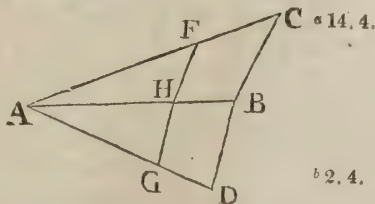


For, the triangles $\triangle ADC$, $\triangle EHG$, and $\triangle BDC$, $\triangle FHG$ as well as the wholes $\triangle ABC$, $\triangle EFG$ being equiangular ^{7. Hyp.}
 Thence is ^{14. 4.} $AB : EF :: AC : EG$; and ^{and Ax.} $CD : GH$;
 and $AD : EH :: DC : HG$; ^{7.} $BD : FH$.

THEOREM XXI.

If, in two triangles ($\triangle ABC$, $\triangle ABD$) having one side (AB) common to both, from any point H in that side, two lines (HF , HG) respectively parallel to two contiguous sides (BC , BD) be drawn to terminate in the two remaining sides (AC , AD); those lines (HF , HG) will have the same ratio as the sides (BC , BD) to which they are parallel.

For, $AB : AH :: BC$
 HF^a , and $AB : AH ::$
 $BD : HG^a$; therefore,
 by equality, $BC : HF$
 $: BD : HG$; whence,
 alternately, $BC : BD ::$
 HF, HG^b .



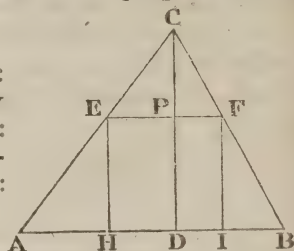
COROLLARY.

Hence, if $BC = BD$, then also will $HF = HG$.

"THEOREM A."

The side (EF) or (EH) of a square (EFIH) inscribed in a triangle (ABC) is to the base of the triangle (AB) as the perpendicular (CD) is to the sum of the base, and perpendicular (AB + CD).

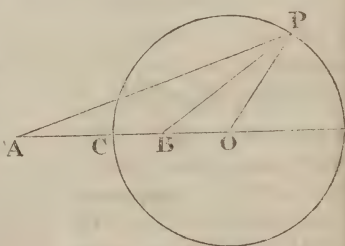
- For EF (PD) : CP :
^a 20. 4. AB : CD^a. Therefore, by
 composition, EF : CD : :
^b Schol. 12. AB : AB + CD^b, and con-
 sequently EF : AB : : CD :
^c 2. 4. AB + CD^c."



"THEOREM B."

If a right line, (AB), be unequally divided in (C), and in it, produced, there be taken (CO) to (AC), as (BC) to (AC - CB); and upon (O), as a center, at the distance of (OC), a circle (CPD) be described, and two lines (AP, BP), be drawn from (A, and B), to meet any where, in the periphery of that circle, then will those lines (AP, BP,) be every where in the constant ratio of (AC to BC.)*

- Draw the radius OP;
 then since CO : AC : :
^a Hyp. BC : AC - CB^a there-
 fore, by composition
 CO : AO : : BC : AC^b;
^b Schol. whence, by alternation
 CO : BC : : AO : AC;
 therefore, by division
 CO : BO : : AO : CO^b,
 or, (by Ax. 2 and 3)



* R. Simson's *Apollonius, or Plane Loci*, prop. 2. Book II.
 —See also Galileo's *Discorsi e Dimostrazioni Matematiche*, 4to, Leyden 1638, p. 39....46, or the English Translation by Weston, Lond. 1730.—Ludlam's *Rudiments of Mathematics*, App. to *Remarks on Euclid*; and Playfair's *Geometry* at the end of Book VI.
 —EDITOR.

$PO : BO :: AO : PO$; wherefore seeing the sides about the common angle, O , are proportional, the triangles BOP , POA , are equiangular^c; and therefore the other sides are proportional, that is $PO (CO) : AO :: PB : AP$ ^d; whence by equality $BP : AP :: BC : AC$.

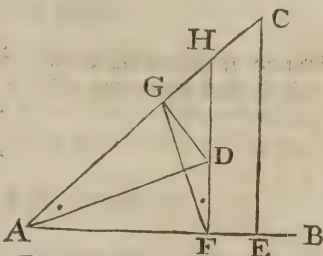
Q, E. D.

THEOREM XXII.

If, at any two points (F, G) in two lines (AB, AC) meeting each other, two perpendiculars (FD, GD) be erected, so as to meet each other; the distance (AL) of their concourse from that of the proposed lines, will be to the distance (FG) of the two points themselves, in the ratio of one of the said lines (AC) to a perpendicular (CE) falling from the extreme of (AC) upon the other (AB).

Let FD be produced to meet AC in H .

Since the angles AFD and AGD are right-ones^c, the circumference of a circle will pass through all the four points A , F , D , G ^d; and so the angles GFD , GAD , standing on the same subtense GD , will be equal^e; and consequently the triangles AHD , FHG equiangular^f; there-fore $AD : FG :: AH : HF$ ^g :: $AC : CE$.



^c Hyp.

^d 19. 3.

^e 11. 3.

^f Cor. 1. to

10. 1.

^g 14. 4.

THEOREM XXIII.

If through any point (P) in a triangle (ABC) three right-lines (AE, BF, CD) be drawn, from the angular points to cut the opposite sides, the segments (AD, BD) of any one side (AB) will be to each other, as the rectangles (AF $\times$ CE, BE $\times$ CF) under the segments of the other sides taken alternately.*

Let GCH be parallel to AB, and let AE and BF be produced to meet it in H and G.

It is manifest, that the triangles FBA, FCG; EAB, EHC; APB, GPH, are equiangular^h.

^a 7. and 3.
of 1.
ⁱ 14. 4.

Therefore $AF : CF :: AB : CG^i$,
 $CE : BE :: CH : AB^i$.

^k 11. 4.
^l 7. 4.
^m 20. 4.

Whence $AF \times CE : CF \times BE :: CH \times AB : CG \times AB^k :: CH : CG :: AD : BD^m$; therefore, by equality, $AF \times CE : CF \times BE :: AD : BD$.

COROLLARY.

Hence, if $AD = BD$, then also will $AF \times CE = CF \times BE$, and therefore $AF : CF :: BE : CE^n$.

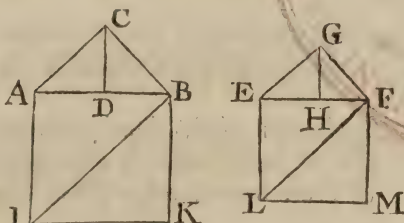
ⁿ 10. 4.

THEOREM XXIV.

Equiangular triangles (ABC, EFG) are in proportion to one another, as the squares (AK, EM) of their homologous sides.

* *Prize Quest. Ladies' Diary, 1735.*

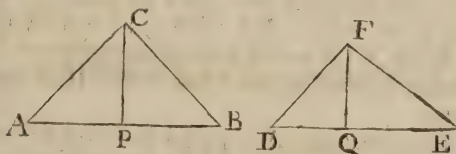
Upon AB and EF let fall the perpendiculars CD and GH, and let the diagonals BI, EL be drawn.



Because $ABC : ABI :: CD : AI$ (AB)^o :: $GH^p : EL^p$ 8. 4.
 $EF (EL) :: EFG : EFL^o$; therefore, alternately, ^p 20. 4.
 $ABC : EFG :: ABI : EFL^o :: AK : EM^r$. ^q 2. 4.
^r Cor. 2. to 2. 2.

THEOREM XXV.

Triangles (ABC, DEF) having one angle (A) in the one, equal to one angle (D) in the other, are in the ratio of the rectangles (AC × AB, DF × DE) contained under the sides including the equal angles.



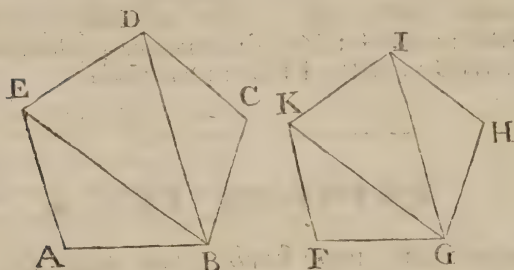
Upon AB and DE, let fall the perpendiculars CP and FQ. Then $AC \times AB : CP \times AB :: AC : CP^s :: DF : FQ^s$ 7. 4.
 $CF : FQ :: DF \times DE : FQ \times DE$; whence, alternately, ^s 14. 4.
 $AC \times AB : DF \times DE :: CP \times AB : FQ \times DE ::$
triangle ABC : triangle DEF^u. ^u Cor. 1. 4.

COROLLARY.

Hence, if the rectangles of the sides containing the equal angles, be equal, or the sides themselves reciprocally proportional^w, the triangles will be equal. ^w 10. 4.
The same also holds in parallelograms, being the doubles of such triangles.

THEOREM XXVI.

All similar right-lined figures (ABCDE, FGHIK) are in proportion to one another as the squares of their homologous sides (AB, FG.)



Draw the right-lines BE, BD, GK, GI.

^z Def. 14.
of 4.

^y 15. 4.

^x Ax. 5.
of 1.

^w 14. 4.

Because $A = F$, and $AB : AE :: FG : FK^z$, the triangles BAE, GFK are equiangular^y; therefore, if from $AED = FKI^x$, there be taken $AEB = FKG$, the remainders BED, GKI will also be equal^z. Wherefore, since $ED : KI (:: EA : KF^x) :: EB : KG^w$, the triangles EBD, KGI are likewise equiangular^y. In the same manner it will appear, that DBC, IGH are also equiangular.

^a 24. 4.

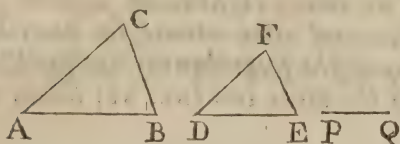
Therefore, because $ABE : GFK (::^a BE^2 : GK^2) : BED : GKI (::^a BD^2 : GI^2) ::^a BDC : GIH$, it is evident, that the sum of all the antecedents (ABCDE) is to the sum of all the consequents (FGHIK) as the first antecedent ABE is to the first consequent GFK^b, or as AB^2 to FG^2 ; which was to be demonstrated.

^b Cor. to
6. 4.

THEOREM XXVII.

If three right-lines (AB, DE, PQ) are proportional, the right-lined figure (ABC) upon the first, will be in proportion to the similar, and

similarly described, figure (DEF) on the second, as the first line (AB) to the third (PQ.) See Def. 7.



For, $AB : PQ :: AB^2 : AB \times PQ^c$ ($^d DE^2$) $:: ^c 7. 4.$
 $ABC : DEF^e.$

^{d} Hyp.
 and 10.
 of 4.
 ^{e} 26. 4.

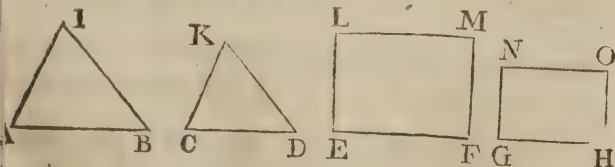
COROLLARY.

Hence, similar right-lined figures, are in the duplicate ratio of their homologous sides ^{f} .

^{f} Def. 7.
 of 4.

THEOREM XXVIII.

If four right-lines (AB, CD, EF, GH) be proportional, the right-lined figures described upon them; being similar, and similarly situated, shall also be proportional (ABI : CDK :: EM : GO.)

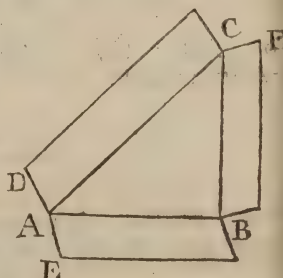


For, $ABI : CDK (:: ^g AB^2 : CD^2 :: EF^2 : GH^2^h)$ ^{g} 26. 4.
 $: EM : GO^g.$ ^{h} Cor. 1. to 11. 4.

THEOREM XXIX.

If upon the three sides of a right-angled triangle (ABC) as many right-lined figures (CD, BE, BF) like, and alike situate, be described, that (CD) upon the hypotenuse (AC) will be equal to both the other two (BE, BF) taken together.

For, $BE : BF :: AB^2 :$
¹ 26. 4. BC^2 ; therefore (by com-
 position) $BE + BF : BE ::$
^m 8. 2. $AB^2 + BC^2 (=^m AC^2) : AB^2$
ⁿ Ax. 3. $:: CD : BE^1$; and con-
 of 4. sequently $BE + BF = CD^n$.



END OF BOOK THE FOURTH.

ELEMENTS

OF

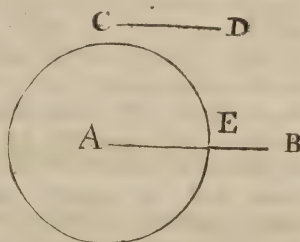
GEOMETRY.

BOOK V.

PROBLEM I.

FROM the greater (AB)
of two unequal lines
(AB, CD) to cut off, or
take away a part (AE)
equal to the less (CD.)

From A as a center, with
radius equal to CD, let
the circumference of a cir-
cle be described^a, cutting^b
AB in E; and the thing is done.

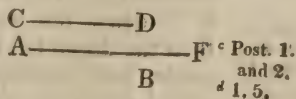


^a Post. 3.
^b Ax. 2.

PROBLEM II.

At a given point (A) to make a line (AB) equal
to a given line (CD.)

Draw the indefinite
line AF^c; from which
take away AB = CD^d;
and the thing is done.



^c Post. 1.
and 2.
^d 1. 5.

PROBLEM III.

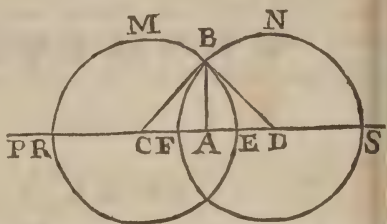
At a given point (A) in an infinite right-line (PQ) to erect a perpendicular.

^e 1. 5.

^f Post. 3.

^g Post. 1.

In the line pro-
pounded, take two
equal distances
AC, and AD^e;
from the centers C^f
and D^g, with any
equal radii greater
than AC (or AD,) let two circles



EMR and FNS be described; which will cut each other: and, from the point, B, of their intersection, draw BA, and the thing is done.

^h Ax. 2.

ⁱ Hyp.

^l Def. 11.
3.

^m Constr.
and Def.
33. of 1.

ⁿ Obs. on
Ax. 10.

^p Constr.
Ax. 10.

For, let the points R, E, and F, S, be those in which the infinite line PQ intersects the circumferences of the two circles EMR and FNS^h: then AF being FD^h (or CRⁱ) $\sqsupset$ AR^h; and AS $\sqsubset$ DS^h (or CEⁱ) $\sqsubset$ AE^h, the point F falls within, and the point S without the circle EMR; and so the two circles cut each other^l. If, therefore, from the point of intersection, BC and BD be drawn; then the triangle CBD being isosceles^m, the angles BCD, BDC at the base will be equalⁿ; and so, CA being = AI, and CB = BD^m, the angle CAB is also = DAB^p.

Otherwise:

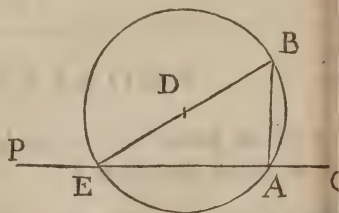
^q Post. 3.

^r Cor. to
4. 3.

^s Post. 1.

^t 13. 3.

From any point D above the line PQ, as a center, through the given point A, let the circumference of a circle be described^q intersecting PQ in



E^r; draw the diameter EDB, and also BA^s; then the angle EAB being in the semicircle^t, is a right angle, and consequently BA is perpendicular to PQ which was to be done.

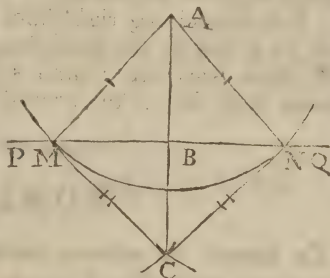
COROLLARY.

From the former of these constructions it appears, that, if from any two points, with two equal radii, greater, each, than half the distance of those points two circles be described : those circles will cut each other.

PROBLEM IV.

From a given point (A) upon an infinite right-line (PQ) to let fall a perpendicular (AB.)

From the given point A as a center, let an arch of a circle be described, as to pass below PQ and intersect it in M and N; from which points, with any equal radii, greater than half MN, let two other arches be described, and from the point of their inter-



section C, let the right-line CBA be drawn; which will be perpendicular to PQ.

For, let AM, AN, CM and CN be drawn; then AM being = AN^u, and MC = NC^w, the angle AMB is = ANB^x, and CMB = CNB^x; and consequently AMC = ANC^y: whence (as AM = AN, and MC = NC) the triangles AMC, ANC are equal in all respects^z; and so, the angle MAB being = NAB, the angle MBA likewise = NBA^z.

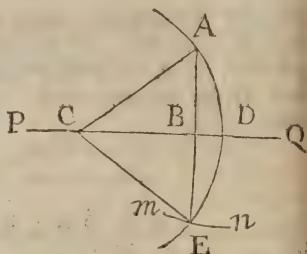
^u Def. 33.
^w of 1.
^x Hyp.
^y and Def. 33. 1.
^z Obs. on Ax. 10.
^u Ax. 4.
^w Ax. 10.

The same otherwise.

^a Post. 3.

^b Constr.

From any point C in the line PQ , as a center, let the circumference of a circle be described through the given point A^a , intersecting PQ in D^b ; and from the center D , with a radius equal to the distance of the points A and D , let another circle mEn be also described, cutting the former ADE in E ; then draw ABE for the perpendicular required.



For, conceiving right-lines to be drawn from C and D , to A and E , the triangles ACE , ADE will be both of them isosceles^b; and so the demonstration is the same with that of the preceding method.

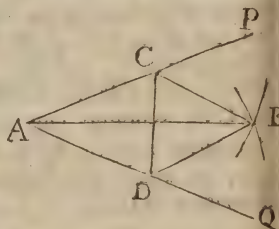
PROBLEM V.

To bisect, or divide into two equal parts, an given right-lined angle (PAQ)

^e 1. 5.

^d Post. 3.

In the lines containing the given angle, take $AC=AD^e$; and upon the centers C and D , with any equal radii, let two circles be described^d, so as to intersect each other; and from the point of intersection E draw EA , and the thing is done.



^e Post. 1.

^f Constr.
and Def.
33. of 1.

^g Ax. 10.

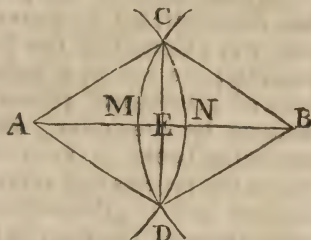
^h Ax. 4.

For let, CD , CE , and DE be drawn^e; then, the triangles ACD and ECD being both isosceles^f, the angle ACD will be $=ADC$, and $ECD=EDC^g$; and consequently the whole angle ACE =the whole angle ADE^h : whence (AC being $=AD$, and $EC=ED$) the angle CAE is also= DAE^g .

PROBLEM VI.

To bisect a given right-line (AB.)

From the extremes A, B, of the given line, with equal radii, describe two circles, so as to cut each otherⁱ; and between the two points of intersection draw CD, cutting AB in E; and the thing is done.



ⁱ Post. 3.
and Cor.
to 3. of 5.

For, if AC, AD, BC and BD be drawn, the triangles ACB, ADB being isosceles^k, thence is the ^k Constr. and Def. 33. of 1.
angle $CAB = CBA^l$, and $DAB = DBA^l$; and conse-
quently $CAD = CBD^m$: whence the triangles ACD ^l Obs. on Ax. 10.
and BCD are equal in all respectsⁿ; and so the angle ^m Ax. 4.
 $ACE = BCE$, $AC = BC$, and CE common, ⁿ Ax. 10.
hence is AE also $= BE^n$.

COROLLARY.

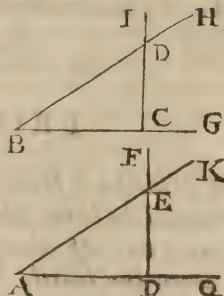
Hence, it is manifest, that CD not only bisects AB, but is also perpendicular to itⁿ.

ⁿ Ax. 10.
and Def.
8. of 1.

PROBLEM VII.

From a given point (A) in a given right-line (AQ) to draw a line (AK) which shall make with the former an angle, equal to an angle given (HBG.)

In BG and AQ take two equal distances BC, AD^o; and at C and D erect the two perpendiculars CI and DF to BG and AQ^p; and in DF take DE equal to the part CD of the former, intercepted by the lines containing the given angle HBG^q; then through E draw AEK^r, and the thing is done^s.



^o 2. 5.

^p 3. 5.

^q 1. 5.

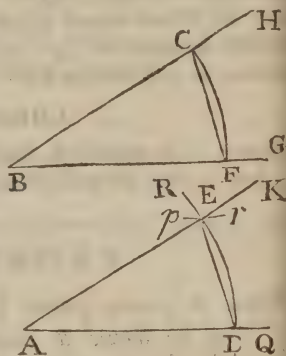
^r Post. 1.

^s Ax. 10.

SCHOLIUM.

Having, in the seven preceding problems, effected and demonstrated, *by means of the axioms only*, whatever was assumed in the fourth postulate, *as barely possible*: we are now authorized, by the most rigid laws of geometrical reasoning, to make use of any theorem or conclusion, whatsoever, derived in the preceding books, in virtue of those assumptions, by which the process and result can be rendered the most obvious and eligible.—Accordingly, by having recourse here to Theorem XIV, of the first book, we shall be able to arrive at a construction of the last problem, better adapted to practice than that above laid down.

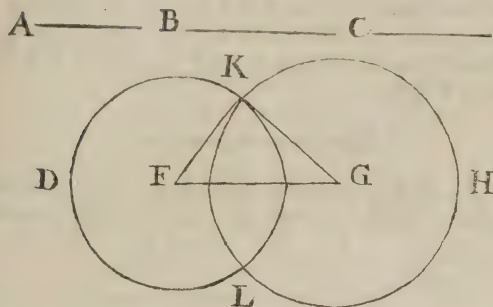
From the centers A and B, at any equal distances AD, BF, let two arcs of circles, FC, DR, be described, ^{Post. 3.} intersecting the given lines in D, F, and C; also from D, with a radius equal to the distance of the points F, C, let another circular arch *pEr* be described, cutting the former DR, in E; then draw AEK, *and the thing is done.*



For, conceiving right-lines to be drawn from F to C, and from D to E, the triangles BFC and ADE will be equilateral to each other, ^{14. 1.} *by construction*, and therefore equiangular also.

PROBLEM VIII.

To describe a triangle, whose three sides shall be equal to three given lines (A, B, C;) provided any two of them, taken together be greater than the third.



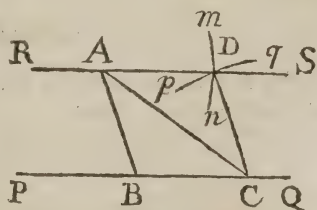
Make $FG = B^w$, and from the centers F and G , with $^w 2.5.$ the intervals, or distances A and C , let two circles DKL , HKL be described ^{x} ; which will cut each ^{x} Post. 3. other; and if, from the point of intersection K , the lines KF and KG be drawn, then will FKG be the triangle required.

For, the distance FG of the two centers, is less than the sum of the radii A and C ; and greater than their ^{y} Ax. 6. 1. difference (because $B + A$ being $\square C$; thence is $B \square$ ^{z} Hyp. $C - A$); therefore the two circles cut each other ^{a} : $^a 9.3.$ consequently $FK = A$, $FG = B$, and $GK = C$. ^{b} Def. 33. of 1.

PROBLEM IX.

Through a given point (A) to draw a right-line (RS) parallel to a given right-line (PQ .)

At any point B in the given line, make BC equal to the distance of the points A and B ; and from the centers A and C , with an interval equal to CB , let 2 circles be described ^{d} , then through their intersection D , let the line RS be drawn, and the thing is done.



$^e 2.5.$

^{d} Post. 3.

Let AB and AC be drawn ^{e} . It is plain, that the ^{e} Post. 1. two circles will cut each other ^{f} , because the sum of $^f 9.3.$

- ^e Constr. their semi-diameters ($=AB+BC$ ^g) is greater than
^h 19. 1. AC ^h: therefore, if ADS and CD be also drawn, then
ⁱ Ax. 1. will $AB=BC=CD=DA$ ⁱ, and therefore RS will be
^k 25. 1. parallel to PQ ^k.

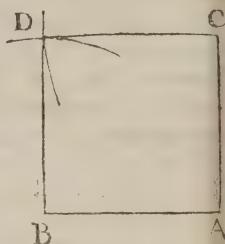
The same otherwise.

- ^l Post. 1. From A , to any point in PQ , draw AB ^l; make the
^m 7. 5. angle $SAB=PBA$ ^m: and then AS will be parallel to
ⁿ 8. 1. PQ ⁿ.

PROBLEM X.

Upon a given line (AB) to describe a square ($ABCD$.)

- Make AC perpendicular, and
^o 3. 5. and equal to AB ^o; and from the
1. 5. centers B and C , let two circles,
with the radius AB or AC , be
^p Post. 3. described ^p, intersecting each
^q 9. 3. other ^q in D ; from which point
draw DB and DC , and the
thing is done.



- For, all the four sides being
^r 25. 1. equal, by construction, the figure is a parallelogram
^s Constr. and therefore the angle A being a right-angle ^s, the
^t Cor. to other three will be all right-angles ^t, and $ACDB$
24. 1. square ^u.
^u Def. 26.

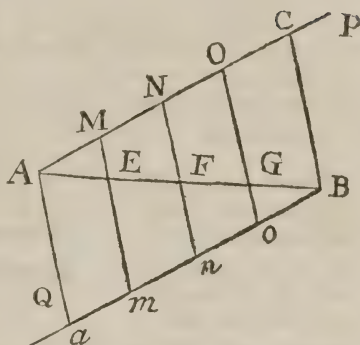
SCHOLIUM.

By the same method a rectangle may be described the sides being given.

PROBLEM XI.

To divide a given line (AB) into any proposed number of equal parts.

From the extremes of the given line AB, draw two indefinite lines AP, BQ parallel to each other^v; in each of these lines let there be taken as many equal distances AM, MN, NO, OC; Bo, on, nm, na (of any length at pleasure) as you would have AB divided into^w; then draw Mm, Nn, Oo,

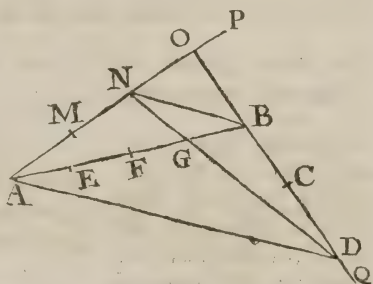


intersecting AB in E, F, G, and the thing is done.

For MN and mn being equal and parallel^x, FN will^z Constr. be parallel to EM^y; and in the same manner will GO^y 26. 1. be parallel to FN: therefore, AM, MN, NO, &c. being all equal^x, AE, EF, FG, will likewise be equal^z. ^z Cor. 1. 1. to 27. 1.

The same otherwise.

In any right-line P, drawn from A, take as many equal distances (AM MN, NO) wanting one, as you would have AB divided into; then, having drawn the indefinite line OBQ, in it take an equal number of parts or distances OB, BC, CD, each of the length of OB, and let DN be drawn, cutting AB in G; make GF, FE, each equal to BG, and the thing is done.

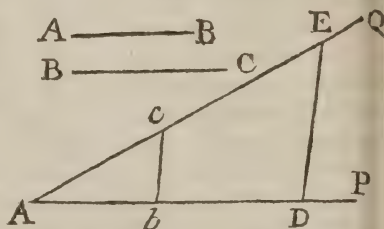


For, if AD and BN be drawn, they will be parallel^a, ^a Cor. to (because OA : ON :: OD : OB^b;) and so, the triangles BNG, ADG, being equiangular^c, it will be BG : AG :: BN : AD^d :: ON : OA^d. Therefore BG is the same part of AG, as ON is of OA. ^{12. 4.} ^b Constr. ^c 3. and ^d 7. 1. ^d 14. 4.

PROBLEM XII.

To two given lines (AB , BC) to find a third proportional.

- From any point A , draw two indefinite lines AP , AQ , in which take $Ab = AB$, $Ac = BC$ and $bD = BC$; draw bc , and parallel to bc , draw DE cutting AQ in E ; then cE will be the third proportional required:



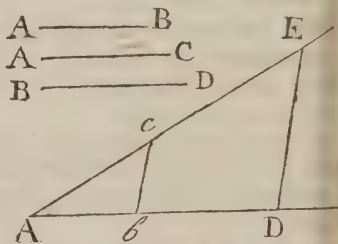
12. 4.

For $Ab (AB) Ac (BC) :: bD (BC) : cE$.

PROBLEM XIII.

To three given lines (AB , AC , BD) to find fourth proportional.

- Having drawn AP and AQ , as in the preceding problem, take therein $Ab = AB$, $Ac = AC$, and $bD = BD$; draw bc , and parallel to it, draw DE , intersecting AQ in E ; then is cE the fourth proportional required.



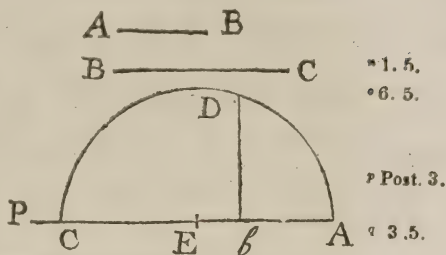
12. 4.

For, $Ab (AB) : Ac (AC) :: bD (BD) : cE$.

PROBLEM XIV.

Between two given lines (AB , BC) to find a mean proportional.

In the indefinite line AP, take $Ab = AB$, and $bC = BC$; bisect AC in E, and from the center E, at the distance of EA, or EC, let a semi-circle ADC be described; erect bD, perpendicular to AC, cutting the circumference in D; then will bD be the mean proportional required.



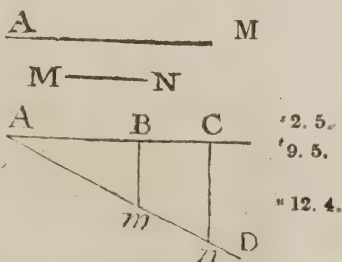
For, $Ab (AB) : bD : : bD : bC (BC)$.

19. 4.
and it's
Corol.

PROBLEM XV.

To divide a given line (AC) into two parts (AB, BC) having the same proportion as two given lines (AM, AN).

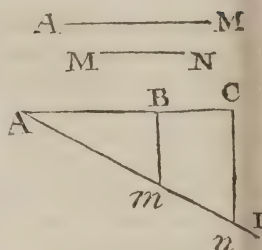
From A draw AD, making any angle with AB; in which take $Am = AM$, and $mn = AN$; draw nC , and mB parallel thereto, meeting AC in . Then will $AB : BC :: m (AM) : mn (MN)$. Which was to be done.



PROBLEM XVI.

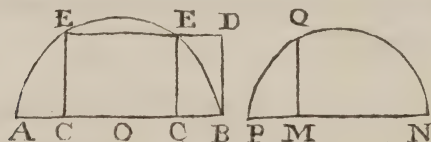
To add a line (BC) to a line given (AB) so that the whole compounded line (AC) shall be in proportion to the part added, as one given line (AN) is to another (MN).

From A draw AD, making any angle with BA; in which take $An = AN^w$, and $nm = NM$; draw mB and nC parallel thereto^z, meeting AB produced, in C; then will $AC : BC :: An (AN) : mn^y$ (MN). Which was to be done



PROBLEM XVII.

To divide a given line (AB) into two such parts (AC, BC) that the rectangle contained under them, shall be equal to the rectangle under two given lines (PM, MN); provided that the given rectangle is not greater than the square of half the line (AB) to be divided.



Between PM and MN take a mean-proportionals MQ^z ; make BD perpendicular to AB, and equal to MQ ; bisect AB in O^a , from which, as a center, let semi-circle be described; and draw DE parallel to BA^b, which (because BD is less than the radius^c) will meet the circle in some point E; from which, upon AB let fall the perpendicular EC: so shall $AC \times B =^d EC^2 = DB^2 (=QM^2) = PM \times MN^e$. Which was to be done.

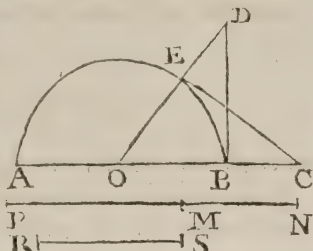
^z 14. 5.
^a 6. 5.
^b 9. 5.
^c Hyp.
^d Cor. to 19. 4.
^e Hyp. and 10. 4.

PROBLEM XVIII.

To a given line (AB) to add another line (BC) such that the rectangle under the whole

compounded line (AC) and the part added, shall be equal to a rectangle under two given lines (PM, MN).

Between PM and MN
take a mean-proportional
RS^f; make BD perpendicular
to AB^g, and equal
to RS^h; bisect AB in Oⁱ,
draw OD, and take OC
= OD^k: so shall AC ×
BC = PM × MN, as was
to be done.



^f 14. 5.

^g 3. 5.

^h 2. 5.

ⁱ 6. 5.

For, if through A and B, from the center O, the
circumference of a circle be described, cutting DO in
E, and E, C be joined; then, the triangles OCE, ODB
(having OE = OB, OC = OD^k, and the angle EOB
common) will be equal in all respects^l; and so, EC
being a^m tangent to the circle in E, we haveⁿ AC ×
BC = CE² =^o BD² = RS² = PM × MN^p.

^k Const.

^l Ax. 10.

^m 6. 3.

ⁿ Cor. to

22. 3.

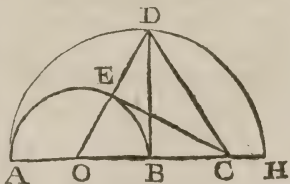
^o 1. 2.

^p 10. 4.

PROBLEM XIX.

*To divide a given line (BH) into two such parts,
that the square of the one (BC) shall be equal
to the rectangle under the other (CH) and a
second given line (AB).*

Taking BA in the same
straight line with BH, be-
tween them let a mean
proportional BD be found^q;
bisect AB in O^r; Draw OD,
and make OC = OD; so
shall BC² = CH × AB, as was
to be done.



^q 14. 5.

^r 6. 5.

For by the demonstration of the precedent, AC ×
C (= CE² = BD²) = AB × BH; from each of which
taking away AB × BC, there remains BC² = AB
× CH^s.

^s 5. 2. and

Ax. 5. 1.

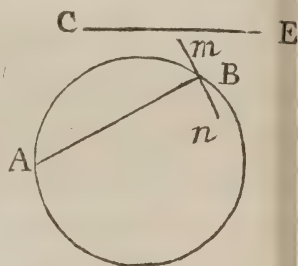
COROLLARY.

If $AB=BH$, then will $BC^2=BH \times CH$; in which case the line BH is said to be divided according to *extreme and mean-proportion*.

PROBLEM XX.

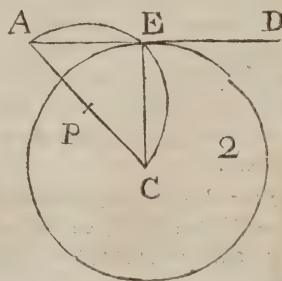
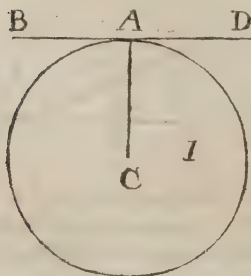
In a given circle, to apply, or inscribe a line (AB) equal to a given line (CE,) less than the diameter of the circle.

- From any point A in the circumference, with the radius CE, let a circular arch *mn* be described^t, cutting the given circle in B^u; then draw AB, and the thing is done^w.
- ^t Post. 3.
^u 9. 3.
^w Def. 33. of 1.



PROBLEM XXI.

To draw a tangent to a given circle (c) through a given point (A).



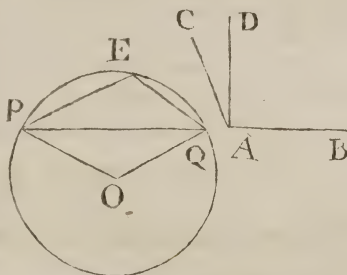
CASE I. *If the given point be in the circumference;*
 then, to the center C, draw AC, and perpendicular to
 C^y draw BAD; which will touch the circle at A^z. ^y 3. 5.
^z 6. 3.

CASE II. *If the point A be without the circumfer-*
ence; then drawn AC, which bisect in P^b; and from ^b 6. 5.
 the center P, at the distance of AP, or CP, let a semi-
 circle AEC be described^c, cutting the given circle in ^c Post. 3.
^d; then draw AED, which will be the tangent ^d 9. 3.
 required^e; because (CE being drawn) AEC is a right-^e 6. 3.
 angle^f. ^f 13. 3.

PROBLEM XXII.

*Upon a given line (PQ) to describe a segment of
 a circle (PEQ) to contain an angle (E) equal
 to a given angle (BAC).*

Make AD perpendi-
 cular to AB^g; also
 make PQO, and QPO,
 each, equal to DAC^h
 (the difference between
 the given angle and a
 right one); then upon
 the point of intersection
 O, as a center, at the
 distance of OP (or OQ)
 let a circle be described; and the thing is done.



^g 3. 5.

^h 7. 5.

For the angle $E = \text{right-angle} + QPO^i = DAB +$ ⁱ 16. 3.
 $\angle DAC^k = BAC.$

^k Constr.

SCHOLIUM.

In the same manner the problem may be con-
structed, when the given angle is acute; only the
lines PO, QO must then be drawn on the other side
of (PQ) as is manifest from the 16th theorem of the
31 book.

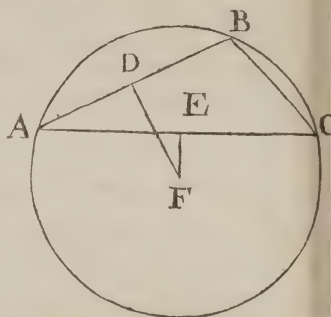
PROBLEM XXIII.

About a given triangle ($\triangle ABC$) to describe a circle

Let any two sides, AB and AC , be bisected by two perpendiculars DF and EF ^k; which will intersect each other in the center (F) of the required circle^l; from whence the circle may be described.

^k 6. 5.

^l Proof of
18. 3.



SCHOLIUM.

By the same method, the circumference of a circle may be described through any three given points, not situated in the same right-line:

Also from hence the center of a circle may be found by having the segment of a circle given.

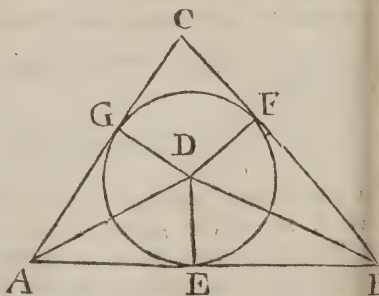
PROBLEM XXIV.

To inscribe a circle in a given triangle ($\triangle ABC$).

Bisect any two of the angles, A and B , by the lines AD and BD ^m, meeting each other in D ; make DE perpendicular to AB ⁿ; then, if from the center D , at the distance of DE , a circle be described, it will touch all the sides of the triangle.

^m 5. 5.

ⁿ 4. 5.



For, let DG and DF be perpendicular to AC and C^p ; then the triangles ADE, ADG, having two ^{p 4. 5.} angles equal, each to each (by construction) and AD common, will not only be equiangular ^{7 Cor. 1.}, but also have ^{to 10. 1.} $DE=DG$. By the same argument $DE=DF$; ^{15. 1.} therefore the circumference of the circle also passes through G and F ⁸; but it touches the sides of the ^{7 Def. 33.} triangle in those points ^{of 1.}, because G and F are right- ^{6. 3.} angles ⁹. ^{9 Constr.}

PROBLEM XXV.

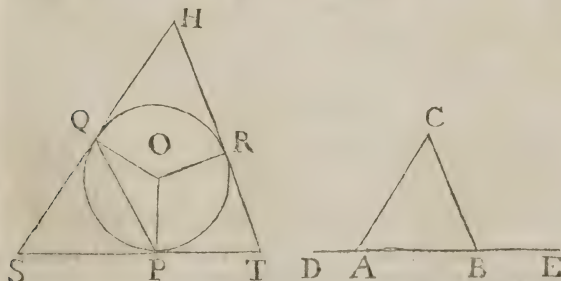
In a given circle (AFB) to describe a triangle, equiangular to a given triangle (PQR).

From the center C, draw the radii CA, CE, CB, making the angles ACE and BCE equal, each, to the angle R ¹⁰; join A, B, and make the angle $BF=Q$ ¹⁰. and from the point F, where BF cuts the circle, draw FA; so shall AFB be the triangle required. ^{7. 5.}

For, $ABF=Q$ ¹⁰, $F (=ACE)=R$ ¹⁰; and consequently $BAF=P$ ¹⁰. ^{10 Constr.}

PROBLEM XXVI.

About a given circle (O,) to describe a triangle, equiangular to a given triangle (ABC).



Produce out the side AB both ways; and draw the radii OP, OR, OQ, so as to make the angle POI = EBC, and POQ = DAC^a; then draw three right lines to touch the circle in the points P, Q and R^b and the thing is done.

For, if PQ be drawn, the angles SQP and SPQ, will be less than the two right-angles SQO and SPO^c and so PS and QS, not being parallels^d, they will meet each other^d; therefore, as the like may be inferred with regard to PT and RT, &c. it is manifest that the three tangents form a triangle STH. Now POR + T being = two right-angles^e = ABC + EBC. and POR = EBC^g; thence will T = ABC: and, by the same argument, S = BAC; whence also H = C.

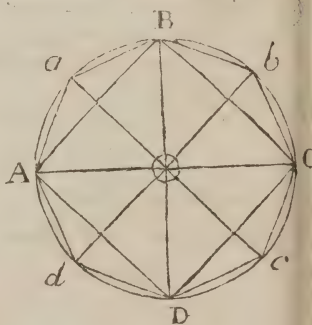
^a 7. 5.
^b 21. 5.
^c Ax. 2. 1.
and 6. 3.
^d Cor. 2.
to 7. 1.
and
Cor. to
5. 1.
Cor. 2.
to 11. 1.
^f 1. 1.
^g Constr.

PROBLEM XXVII.

In a circle given (ABCD) to inscribe a square.

Draw two diameters AC and BD perpendicular to each other^h; then draw AB, BC, CD and DA; so shall ABCD be a square inscribed in the circle.

For, the angles AOB, BOC, DOC and DOA (as well as the sides OA, OB, OC, OD, containing them) being equalⁱ, the opposite sides AB, BC, CD, DA will likewise be equal^k: and the angles ABC, BCD, CDA, DAB, are all of them right-angles^l, and therefore are equal.



ⁱ Ax. 7. 1.
and Def.
33. 1.

^k Ax. 10.
of 1.

^l 13. 3.

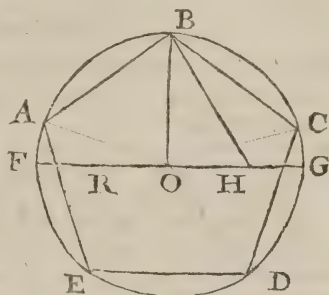
SCHOLIUM.

If two other diameters *ac*, *bd* be drawn (by prob 5.) to bisect the angles AOB, BOC, a regular octagon *Aa Bb Cc Dd* may be inscribed in the circle. And if all the angles at the center O, be again bisected, a regular polygon of sixteen sides, may in like manner be determined; and so on, at pleasure.

PROBLEM XXVIII.

In a circle given (ABGE) to inscribe a regular pentagon

At the center O, upon the diameter FG, erect the perpendicular OB^m, meeting the circumference in B; divide OG in H (by prob. 19.) so that $OH^2 = GH \times OG$; that is, take $OR = \frac{1}{2}OF$, and $OH = \text{dist. } RB$: Then draw BH; which will be equal to the side of the pentagonⁿ; from whence the figure itself may be described.



m 3. 5.

described. *28. 3. and 8. 2.

SCHOLIUM.

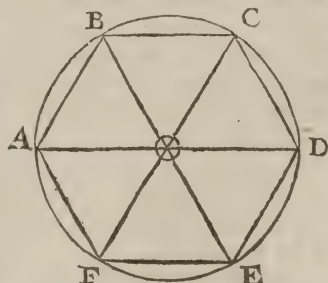
Hence a regular decagon may be inscribed in the circle; the side being = OH^o.

° 28. 3.

PROBLEM XXIX.

In a circle given, to inscribe a regular hexagon (ABCDEF.)

From the extremes of any diameter AD, apply AB, AF, DC, and DE equal, each, to the radius OP; then join B, C, and E, F; and the thing is done.



p 20. 5.

For, if the radii OB, OC, OE, OF be drawn; the triangles AOB and DOC, being equilateral^q, will also have the angle ^q Constr. $\angle AOB = \angle DOC$; whence AB is parallel (as well as ^r 14. 1. equal) to OC^s; and consequently BC and AO are ^r Cor. to 8. 1.

'26. 1. likewise equal and parallel': Therefore, seeing the triangles AOB, BOC, COD, &c. are equilateral, and alike in all respects; not only the sides, but also the angles ABC, BCD, &c. of the hexagon, will be equal

* Ax. 4. 1. among themselves".

COROLLARY.

Hence it appears, that *the side of a regular hexagon, inscribed in a circle, is equal to the semi-diameter, or radius.*

SCHOLIUM.

Besides the figures constructed in the preceding problems, and those arising from thence by continual bisections, or taking the differences, no other regular polygon can be described, from any known method, *purely geometrical*, that is, by means of right-lines and circles only.

PROBLEM XXX.

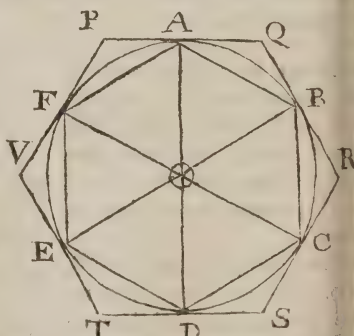
About a given circle to describe a regular polygon, of the same number of sides with a regular polygon (ABCDEF) inscribed in the circle.

From the center O, to the angles of the inscribed polygon, draw OA, OB, OC, &c. and perpendicular thereto draw PAQ, QBR, RCS^t, &c. intersecting^u in P, Q, R, S, T, V; so shall PQRSTV be the polygon that was to be described.

^t 3. 5.
^u Proof of
26. 5.

^w Hyp.

For, by taking away the equal^w angles OAF, OAB, OBA, OBC, &c. from the equal (right) angles OAP, OAQ, OBQ, OBR, &c. the remainders FAP, BAQ, ABQ, CBR, &c. will



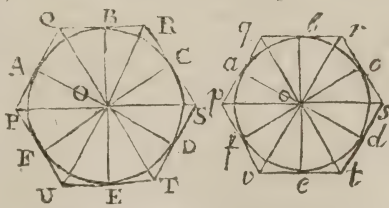
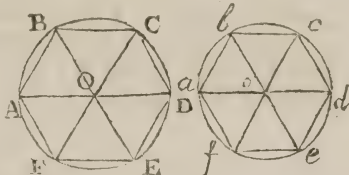
also appear to be equal^z: therefore the triangles FAP, ^z Ax. 5. 1. BQ, BRC, &c. (having also $FA = w AB = BC$, &c.) are equal in all respects^y; and so the angles P, Q, R, ^y 15. 1. &c. as well as the sides PQ, QR, RS, &c. are equal among themselves^z. ^z Ax. 4. 1.

PROBLEM XXXI.

Any two circles (ACE, ace) being given, to describe a polygon in, or about the one (ace) that shall be similar to any polygon described in, or about the other (ACE.)

First, having drawn the radii OA, OB, &c. the angles of the given inscribed polygon BCDEF; make, at the center o, the angle $ab = AOB$, $boc = BOC^a$, &c. Then, the chords ab, bc, cd , &c. being ^a 7. 5. drawn, I say, the polygon $abcdef$ will be similar to the given one ABCDEF. For, the triangles AOB, ab ; BOC, boc ; &c. being equiangular^b, the angles ^b Cor. 1. ABC, abc must also be equal^c; and ^d $AB : ab (::$ to 10. 1. $B : ob) :: BC : bc$. In the same manner, the other ^c and 12. 1. corresponding angles are equal, and the sides contain- ^c Ax. 4. 1. ^d 14. 4. ing them proportional: Therefore the two polygons are similar^e.

Again, having drawn the radii OA, OB, OC, &c. to the points of contact of the given circumscribing polygon QRSTU; draw likewise the radii oa, ob, oc , &c. making the angle $aob = AOB$, $boc = BOC^f$, ^f 7. 5. &c. perpendicular to which ^g draw pq, qr, rs , &c. so shall ^g 3. 5. the polygon $pqrstv$ be similar to the given one QRSTU. For, the angles OAQ, OBQ, oaq, obq ,



^e Def. 14.
4.

- ^h Ax. 7. 1. being all equal ^h, and AOB also = aob^i ; the remain-
ⁱ Constr. angles AQB, aqb of the two quadrilaterals AOB
^k 11. 1. $aobq$ must be equal ^k; as must likewise their half
 and Ax. OQB, oqb (for the right-angled triangles OAQ, OBQ
 5. having OA = OB, and OQ common, have also OQ
^l 16. 1. = OQB^l.) In the same manner is ORB = orb , &
 whence, the triangles POQ, poq : QOR, qor , &
^m 14. 4. being equiangular, it follows^m that PQ : pq (: : OQ
 oq) : : QR : qr : And so of the rest. Therefo
ⁿ Def. 14. PQRSTU and $pqrstv$ are similarⁿ.
 4.

COROLLARY.

It appears from hence, that *the similar inscribed polygons, as well as the circumscribing ones, are proportion, as the squares of the radii of their respective circles.* For, in the former case $AO^2 : a$
^o 14. 4. : : $AB^2 : ab^2$: : ^p ABCDEF : $abcdef$; and, in the
 and Cor. latter, ^o $AO^2 : ao^2$: : $PO^2 : po^2$: : $PQ^2 : pq^2$: :
 to 11. 4. PQRSTU : $pqrstv$.
^p 26. 4.

THE END OF THE FIFTH BOOK.

ELEMENTS

OF

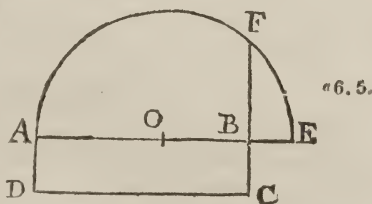
GEOMETRY.

BOOK VI.

PROBLEM I.

To make a square equal to a given rectangle
(ABCD.)

IN one side AB of the rectangle, produced, take BE = the other side BC; and from the center O, at the distance OA, or OE, let a semi-circle AFE be described; and let CB be produced to meet the circumference in F; then a square described on BF (by 10. 5.) will be equal to the given rectangle ABCD^b.



^a6. 5.

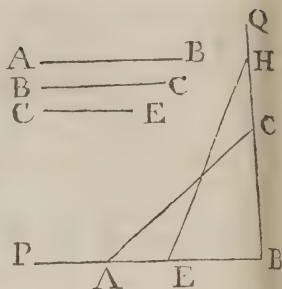
^bCor. to
19. 4.

PROBLEM II.

To make a square equal to the sum of two given squares.

Let AB and BC be the sides of the two given squares.

Draw two indefinite lines BP, BQ, at right-angles to each other^c; in which take BA=BA, BC=BC, and join A, C; then a square described on AC (by 10. 5.) will be equal to the sum of the two squares described upon AB and BC^d.



SCHOLIUM.

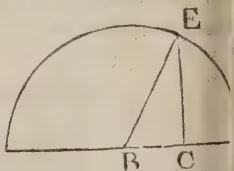
In the same manner a square may be made equal to the sum of three, or more, given squares: for AB, BC, CE be taken as the sides of the given square then, by making BH=AC, BE=CE, and drawing EH, it is evident, that a square upon EH, will equal to the sum of the three squares upon AB, BC and CE; or that, $EH^2 = BH^2 + BE^2 = AB^2 + BC^2 + CE^2$.

PROBLEM III.

To make a square equal to the difference of
given squares.

Let AB and BC (taken in the same strait line) be equal to the sides of the two given squares,

Upon the center B, with the radius BA, let a circle be described, and make CE perpendicular to BC^e, meeting the circumference in E: so shall a square described on CE (by 10. *Cor.* to 4. 1.) be equal to BE² (BA²) − BC².

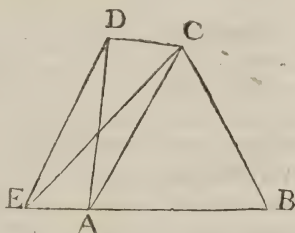


PROBLEM IV.

To make a triangle equal to a given quadrilateral (ABCD.)

Draw the diagonal AC, also draw DE parallel to AC^g, meeting BA produced in E, and join CE; then will the triangle BCE = the given quadrilateral ABCD.

For, the triangles ACE, ACD, being, upon the same base AC, and between the same parallels AC and ED, are equal^h; therefore, if^h Cor. 1. BC be added to each, then also will BCE = ^{to 2. 2.} ABCDⁱ. ^{Ax. 4. 1.}

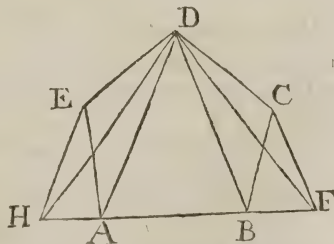


^g 9. 5.

PROBLEM V.

To make a triangle equal to a given pentagon (ABCDE.)

Draw DA and DB, and also EH and CF parallel to them^k, meeting AB produced in H and F; then draw DH and DF; shall the triangle HF = the pentagon BCDE.



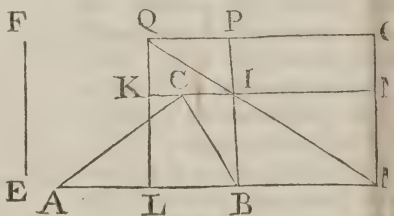
^k 9. 5.

For the triangle DHA is = DEA^l, and DFB = ^{Cor. 1.} CB^l; therefore DHF (= DHA + DAB + DFB = ^{to 2. 2.} EA + DAB + DCB) = ABCDE^m. ^m Ax. 4. 1.

PROBLEM VI.

Upon a given line (EF), to make a rectangle equal to a given triangle (ABC).

Through C, the vertex of the triangle, draw KN parallel to the base ABⁿ; and bisect AB with the perpendicular LQ^o, meeting KN in K; also draw BP perpendicular to ABⁿ, intersecting KN in I; then in AB, produced, take BM=EF, and draw MIQ cutting LQ in Q; draw QO and MO parallel to AM and LQⁿ, meeting each other in O; then will INOP be the rectangle required.

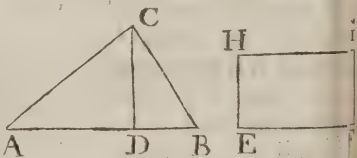


For, it is evident, that LI, IO and LO are the diagonals of the rectangles^o: therefore $IN = BM = EF$ ^s, and $IO = LI$ ^t = ABC^u.

^o Cor. to 24. 1.
^r 24. 1.
^s Constr.
^t 3. 2.
^u Cor. 2. to 2. 2.

The same otherwise.

From the vertex C, upon the base AB, let fall the perpendicular CD^w; make EH perpendicular to EF^x, and equal to a fourth proportional to 2EF. AB, and CD^y: then the rectangle EG contained under EF and EH will be equal to the triangle ABC.



For since, by construction, $2EF : AB :: CD : EH$, therefore is $2EF \times EH = AB \times CD$ ^z, and consequently $EF \times EH = \frac{1}{2}AB \times CD = ABC$ ^a.

^z 10. 4.
^a Cor. 2. to 2. 2.

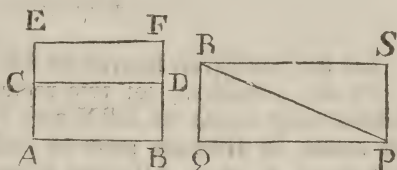
SCHOLIUM.

By either of the two preceding methods, a parallelogram having a given angle may be described upon a given line, equal to a given triangle; if, instead of BP, MLQ, or BDC, FEH being right-angles, you make them all equal to the angle given: the rest of the construction being the same.

PROBLEM VII.

Upon a given line (AB) to describe a rectangle equal to a given right-lined figure (PQRS).

Let the given figure be divided into triangles PQR, PRS: and upon the given line AB (by the precedent) let a rectangle ABDC, equal to the triangle PQR, be described; also upon CD make the rectangle CDEF equal to the triangle PRS: so shall ABEF, which is a rectangle (because both ACE, and BDE are continued right lines^b) be = PQR + PRS^c = PQRS; which was to be done.



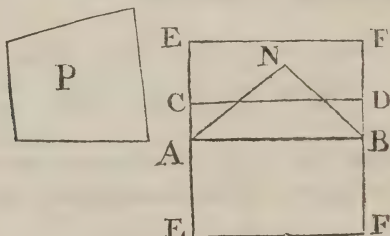
^b 2. 1.
^c Ax. 3.
and 4. 1.

SCHOLIUM.

When the figure given has not more than five sides, the construction will be more easy, by first finding a triangle equal to it (by prob. 4. or 5.), and then making a rectangle equal to that triangle. But if the figure be a rectangle, the easiest way of all, will be to take a fourth proportional BF to the given line AB and the two sides PQ and PS of the given rectangle (7. 13. 5.); which fourth proportional will be the altitude of the rectangle required. For, since A : PQ :: PS : BF (by Constr.) therefore (by 16. 4.) $AB \times BF = PQ \times PS$

PROBLEM VIII.

To describe a rectangle equal to the sum or difference of two given right-lined figures.



Let the two given figures be ABN and P.

By the precedent, let two rectangles AD and AE respectively equal to ABN and P, be described, on the same, or different sides of AB, according as the difference, or sum, of the two figures is required: then will the rectangle CF be equal to that sum or difference^e.

^e Ax. 4.
or 5. 1.

COROLLARY.

Hence *two lines having the same ratio with two given right-lined figures, are determined:*

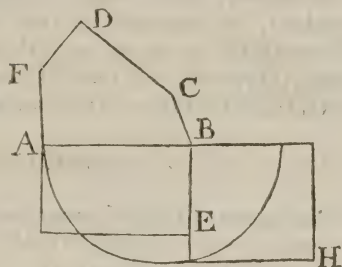
^f 7. 4.

For $AC : AE :: AD \text{ (ABN)} : AF^f \text{ (P)}.$

PROBLEM IX.

To make a square equal to any right-lined figure given. (ABCD F).

Upon AB describe
rectangle AE equal
to ABCD^g; then
make a square BH
equal to that rect-
angle^h, and the thing
is done.

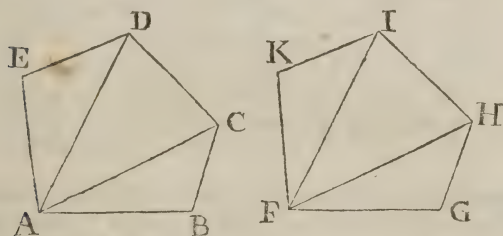


SCHOLIUM.

After the same manner (from prob. 8.) a square
may be described equal to the sum, or difference, of
any two given right-lined figures.

PROBLEM X.

To describe a figure (FGHIK) equal, and similar
to a given right-lined figure (ABCDE).



Draw AC and AD, and also FG equal to ABⁱ; ⁱ 2. 5.
make the angle GFH=BAC^k, HFI=CAD, and ^k 7. 5.
FK=DAE^k; likewise make FH=AC, FI=AD,
and FK=AE^l; then draw GH, HI, and IK, and the ^l Ax. 10.
thing is done. ^{1.}

For, since the triangle FGH=ABC^l and FHI=
ADⁱ, &c.; therefore is the whole polygon FGHIK,
also, equal to the whole polygon ABCDE^m. ^m Ax. 4. 1.

Moreover, these equal triangles being also equiangular^l, it is manifest, that $G=B$, $GHI=BCD^m$, $HIK=CDE$, and so on; therefore, FG being also $=AB$, $GH=BC$, $HI=CD^l$, &c. the two polygons

" Def. 14.
4.

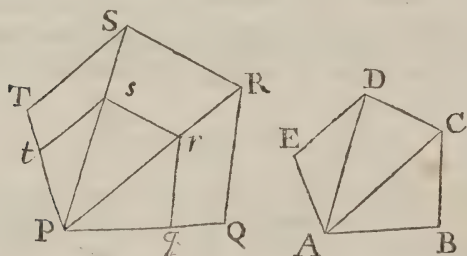
$ABCDE$, $FGHIK$ are similar to each otherⁿ.

SCHOLIUM.

The figure $FGHIK$ may be otherwise constructed, by making the triangles FGH , FHI , &c. respectively equilateral to ABC , ACD , &c. as is evident from 14. 1. and Ax. 4.

PROBLEM XI.

Upon a given line (AB) to describe a figure (ABCDE) similar to a given right-lined figure (PQRST.)



Draw PR and PS , and in PQ (produced, if needful) take $Pq=AB$; draw qr parallel to QR ^o, meeting PR in r ; also draw rs and st , parallel to RS and ST intersecting PS and PT in s and t : then upon AB , (by the precedent), describe a polygon equal and similar to $Pqrst$; and the thing is done.

For, since any angle BCD (qrs) of the polygon $ABCDE$, is equal to its correspondent QRS ¹ and also CB (qr) : CD (rs) :: RQ : RS ², therefore the two polygons $ABCDE$, $PQRST$, are like to each other^r.

¹ Cor. 1.
to 7. 1.
² 21. 4.

^r Def. 14.
of 4.

SCHOLIUM.

This last problem may be otherwise constructed by making the triangles ABC, ACD, ADE equiangular to the triangles PQR, PRS, PST, respectively.

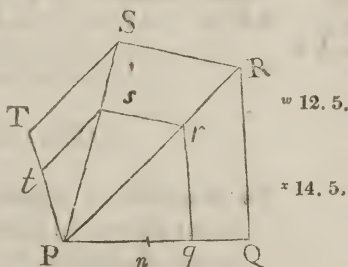
For, then the angle BCD being = QRS, CDE = STS, &c.; and also $BC : QR :: AC : PR :: AD : RS$, &c. the two polygons must therefore be similar to each other^u,

^u Def. 14.
of 4.

PROBLEM XII.

To describe a figure similar to a right-lined figure given (PQRST), which shall be to it in a given ratio of one right-line to another.

In PQ (produced if necessary) take Pn to Pq in the given ratio of the figure to be described to the figure given^w; and in the same line Q, take Pq equal to a mean-proportional between Pn and Q^x; upon which (by the precedent) let Pqrst, similar to PQRST, be described, and the thing is done.

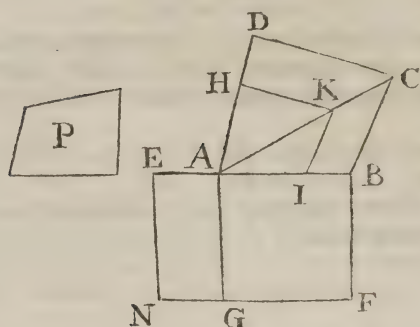


For, since $Pn : Pq :: Pq : PQ$ (by Constr.); therefore $Pn : PQ :: Pqrst : PQRST$ ^y.

^y 27. 4.

PROBLEM XIII.

To describe a figure that shall be equal to one right-lined figure given (P), and similar to another (ABCD).



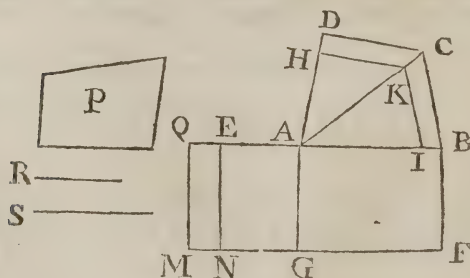
^a 7. 6. Upon AB make the rectangle $ABFG = ABCD^z$, and upon AG make the rectangle $AGNE = P^z$; in AB take AI equal to a mean proportional between AB and AE^a ; and upon AI let $AIKH$ be described similar, and alike situated, to $ABCD^b$, and the thing is done.

^c 7. 4. For $ABCD$ (AF) : P (AN) :: AB : AE^c : : $ABCD$: $AIKH^d$; and therefore $AIKH = P^e$.

^d 27. 4,
^e Ax. 4. 4.

PROBLEM XIV.

To describe a figure similar to a given right-lined figure ($ABCD$). which shall be to another given right-lined figure (P) in a given ratio of one right-line (s) to another (r).



Make the rectangle $ABFG = ABCD^g$, and the rect-^g 7. 6.
 angle $AGNE = P^g$; also in AE , produced, take AQ
 $=$ a fourth-proportional to R, S , and AE^h ; then, (*by the*^h 13. 5.
precedent), make $AIKH$ similar to $ABCD$, and equal
 to the rectangle $AG \times AQ$: then will $AIKH$ ($AG \times$
 AQ) : P ($AG \times AE^i$) :: AQ : AE^k :: S : R^i ; ^{i Constr.}
which was to be done. ^{k 7. 4.}

THE END OF THE SIXTH BOOK.

ELEMENTS OF GEOMETRY.

BOOK VII.

DEFINITIONS.

1. **A RIGHT-LINE** is said to be *perpendicular to a plane*, when it is perpendicular to all right lines that can be drawn in that plane, from the point or which it insists.

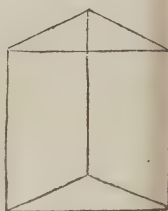
2. One *plane* is said to be *perpendicular to another plane*, when all right-lines drawn in the one, perpendicular to the common section, are perpendicular to the other.

3. *Parallel-planes* are those which are every where equally distant, the one from the other.

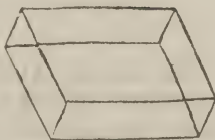
4. A *Solid* is that, which has length, breadth, and thickness.

5. *Similar solids* are such as are bounded by an equal number of similar planes.

6. A *Prism* is a solid, the planes of whose sides are parallelograms, and whose two ends, or opposite bases, are plane, rectilineal figures, parallel to each other.

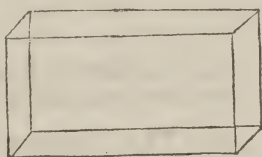


7. A *parallelepipedon* is a solid bounded by six parallelograms, of which the opposite ones are parallel, equal, and like to each other.

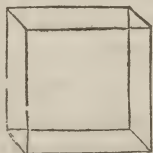


8. An *upright prism*, or parallelepipedon, is that, in which the planes of the sides are perpendicular to the plane of the base.

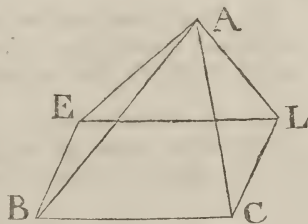
9. A *rectangular parallelepipedon* is that whose bounding planes are all rectangles, and which stand at right-angles one to another.



10. When all the bounding planes are squares, the parallelepipedon is called a *cube*.

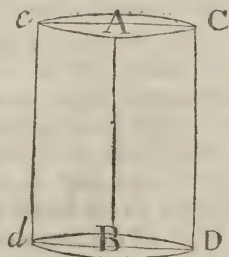


11. A *Pyramid* is a solid, whose base is any right-lined plane figure, and whose sides are triangles, having all their vertices united in a point, above the base, called the *vertex* of the pyramid. B

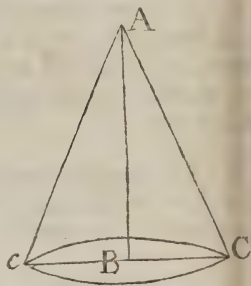


Thus ABCLE represents a pyramid, whose vertex is A, and base BCLE.

12. A *Cylinder* (DCcd) is a solid generated by the rotation of a rectangle ACDB about one of its sides AB, supposed at rest; which quiescent side AB is called the *axis* of the cylinder.



13. A *cone* (ACc) is a solid generated by the rotation of a right-angled triangle ABC about its perpendicular AB , which is called the *axis* of the cone.



14. A *Sphere* is a solid generated by the rotation of a semi-circle about its diameter.

15. The *Frustum* of a pyramid, or cone, is the part which remains when any part next the vertex cut off by a plane parallel to the base, is taken away.

16. The *altitude* of a pyramid, or prism, is the perpendicular distance of the vertex, or upper plane from the plane of the base.

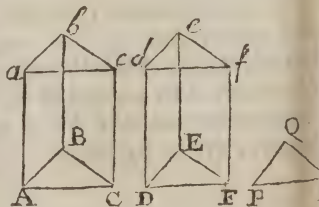
17. Every rectangular parallelepipedon is said to be contained under the three right-lines that are the length, breadth, and altitude.

18. A *Plane* is said to be *extended* (or *to pass*) by a right-line, when every part of the latter is placed in, or touched by the former.

AN AXIOM.

Upright prisms ($AabcCBA$, $Ddef FED$) of the same altitude, standing upon bases (ABC , DEF) equal and like to each other, are themselves equal.

To see the evidence of this AXIOM in the strongest light, conceive a right-lined plane figure PQR to be formed, equal and like in all respects to the bases ABC , DEF of the two prisms:

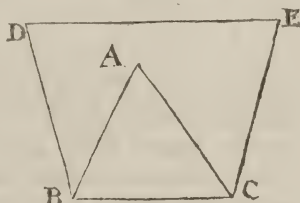


upon which, conceive the prisms to be placed, one after another, so that their bases may coincide therewith. Then, because the planes of the sides stand, in both cases, perpendicular to the plane of the base, upon the same lines PQ, QR, PR, and are carried up to the same height, it is manifest that the bounds of the two solids, when thus placed, have the very same position, and, consequently, that the solids themselves occupying (successively) the same identical space, are equal the one to the other.

A POSTULATE.

That by any two right-lines (AB, AC) meeting in a point, a plane may be extended.

In order better to comprehend the sense and design of this Postulate, let a plane BDEC, extended by the right-line joining the points B and C, be conceived to be revolved about upon that line, till it meets with, or takes in, the point A; then the plane including, in that position, all the three points B, C, and A, it also includes, or is extended by, the right-lines AB, AC, BC, joining those points; which are in the same plane with their extremes (by def. 6. 1.)

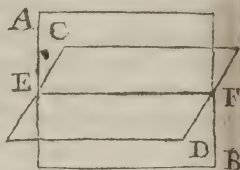


Hence it appears, that by any three points, a plane may be extended; and that all the three sides of any right-lined triangle, are in the same plane.

THEOREM I.

The common section of two planes (AB, CD) is a right-line.

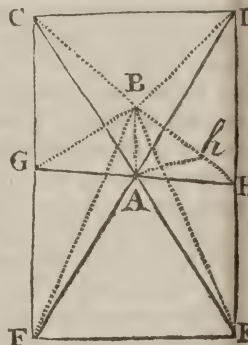
For, between the two extreme points E, F of the common section let a right-line EF be drawn^a; then, that line being in the plane AB^b, and also in the plane CD^b, it must of consequence, be the common section of them both



THEOREM II.

If a right-line (AB) be perpendicular to two other right-lines (CE, DF) cutting each other at the common section (A,) it will be perpendicular to the plane (CDEF) passing by those two lines.

Take AC, AD, AE, AF all equal to one another; and having joined CD, DE, EF, CF, let there be drawn through A, in the plane CDEF, any right-line GH, meeting CF and DE in G and H; and let BC, BG, BF, BD, BH, and BE be also joined.



Because AC^b=AE=AD=AF, and CAF=DAE^c, therefore is CF=DE^d, and the angle FCA (or GCA)=DEA or (HEA); and so, GAC being likewise=HAE^e and AC=AE^b, thence will AG=AH^e and GC=HE.

Again, since the right-angled triangles CAB, DAB, EAB, FAB have their bases all equal^f, and the perpendicular AB common, their hypotenuses BC^d, BD, BE, BF will be equal too; and therefore the triangles CBF, DBE being mutually equilateral, the angle FCB (or GCB) must be=DEB (or HEB^g); whence, GC

^b Constr.

^c 3. 1.

^d Ax. 10. 1.

^e 15. 1.

^f Hyp.

^g 14. 1.

being also $= HE$, and $BC = BE$, thence is $BG = BH^d$:
 Therefore, AG being likewise (as is proved above) $= AH$ and AB common, the angles GAB , HAB are equal, and consequently right-angles^h. In the same^h Def. 8. of 1.
 manner, AB is perpendicular to every other right-line drawn through A in the plane $CDEF$; which was to be demonstratedⁱ.ⁱ Def. 1. of 7.

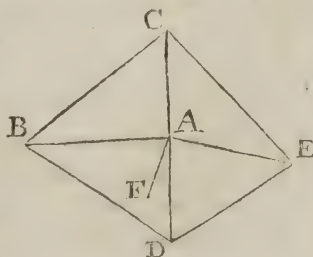
COROLLARY.

Hence it will appear, that, if one right-line (AB) meeting several others (AF , AE , &c.) in the same point (A) is perpendicular to them all; these last will be all in the same plane. Because it is impossible for a right-line (Ah) drawn from A , out of the plane $FEDC$ of the two former of these, to be perpendicular to AB ; seeing the angle BAh is less, or greater, than a right-angle (or BAH^k), according as Ah is^k 2. 7. posited above, or below the said plane $FEDC$.ⁱ Ax. 2. 1.

THEOREM III.

If, through any given point (A) in a given plane (BCD) a line (CD) be drawn, and perpendicular to that line, at the same point (A) two other lines (AB , AE) be also drawn, the one (AB) in the plane given (BCD), and the other in any other plane (CDE) passing by the first line (CD); then I say, that a right-line (AF) drawn from the given point (A) at right-angles to the first perpendicular (AB) in the plane (BAE) of the two, will be perpendicular to the given plane (BCD) at the given point (A).

For CA being perpendicular both to AB and AE^m it will likewise be perpendicular to AF^n ; and so FA being perpendicular to AB^m (as well as to CA) is also perpendicular to the plane BCD , in which AB and CA are drawnⁿ.



^m Hyp.

ⁿ 2. 7.

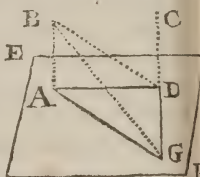
SCHOLIUM.

In this last Theorem, the manner of erecting a perpendicular to a plane, at a point given, is indicated, and the consistence of the *first definition*, of this book, evinced.

THEOREM IV.

Two right-lines (AB, CD) perpendicular to the same plane (EF), are parallel to each other.

Draw in the plane EF, the right-line AD, and also DG perpendicular to AD; make $DG = AB$, and let AG, BG, and BD be drawn.



- The triangles BAD, ADG, having $AB = DG$ ^o, AD common, and the angle BAD (=right-
^o Constr. angle ^p) = ADG ^o, will also have $BD = AG$ ^q: And so
^p Def. 1. 7. the triangles BDG, BAG being mutually equilateral.
^q Ax. 10. 1. the angle ^r BDG must be = BAG = ^p right-angle: But
^r 14. 1. the line CD (as well as BD and AD) being perpendicular to DG ^p, it is therefore in the same plane
^p Def. 1. 7. (CDAB) with BD and AD ^t; and consequently a
^t Cor. to the angles BAD, CDA are both right-ones ^p, it must
^u 4. 1. be also parallel to BA ^u.

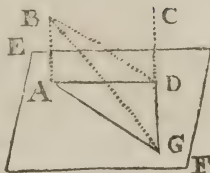
COROLLARY.

Hence, it follows, that *from the same given point to one and the same plane, more than one perpendicular right-line cannot possibly be drawn*: Because all perpendiculars to the same plane, are parallel; but lines drawn from the same point, are not parallels.

THEOREM V.

If, of two parallel right-lines (ABCD) the one (AB) is perpendicular to any plane (EF), the other (CD) shall also be perpendicular to the same plane (EF).

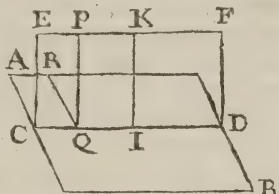
The construction of DG , AG , &c. being supposed the same here as in the preceding Theorem; it appears, from thence, that ADG and BDG are both right-angles: And, because BD , as well as AD , is in the plane of the proposed parallels BA , CD^v , the angle CDG is also a right-^w Def. 6. 1. line^x, as is likewise the angle CDA^y . Therefore CD ^x 2. 7. is perpendicular to the plane ADG^z . ^y 4. 1. ^z 2. 7. and Def. 1.



THEOREM VI.

If a right-line (PQ) be perpendicular to a plane (AB) , any plane (ED) passing by that line, will be perpendicular to the same plane (AB) .

In the plane ED from any point K , draw KI perpendicular to the common section CD . Then, the angle DIK being a right-angle $= DQP^y$, KI will be parallel to PQ^z , and therefore perpendicular to the plane AB^a . By the same inference all other right-lines drawn in the plane ED , perpendicular to the common section CD , are also perpendicular to the plane AB . Therefore, the plane ED itself is perpendicular to the plane AB^b .



COROLLARY I.

Hence it will appear, that the plane AB (according to the sense of the definition) is perpendicular to the plane ED : For a right-line QR drawn in the former, perpendicular to the common section CD , being also ^c perpendicular to PQ , it ^d (and consequently the plane AB in which it is) will be perpendicular to the plane ED^e .

K 2

^c Hyp. an.
^d Def. 1.
^e 2. 7.
6. 7.

COROLLARY II.

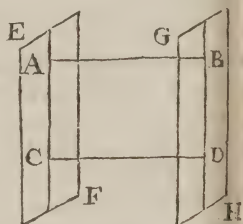
Hence it also appears, that a line standing at right angles to one of two perpendicular planes at any point (I) in the common section (CD) must be in the other plane: For the line IK, in the plane ED, is perpendicular to the plane AB; besides which line, another perpendicular to AB, from the same point I, cannot be drawn^f.

^f Cor. to
4. 7.

THEOREM VII.

Planes (EF, GH) to which one and the same right-line (AB) is perpendicular, are parallel to each other.

From any point C, in the plane EF, let CD be drawn parallel to AB; which (as well as AB) will be perpendicular to both the planes^g; and so the angles A, B, C, D (when AC and BD are joined) being all



^g 5. 7.

^h Def. 1. 7. right-ones^h, the figure ABDC (the sides of which AC, BD

ⁱ Def. 6. 1. are in the same plane with the parallels AB, CDⁱ

^k 4. 1. and will therefore be a rectangular parallelogram^k; and

Def. 24. consequently $CD = AB$ ^l. By the very same argu

^l 24. 1. ment, all other perpendiculars, terminated by the two planes, are equal among themselves; which was to be

^m Def. 3. demonstrated^m.

COROLLARY.

Hence, all right-lines perpendicular to one of two parallel planes, are also perpendicular to the other.

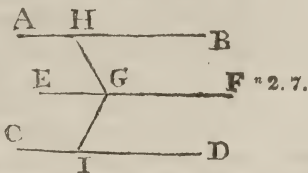
SCHOLIUM.

From the two last Theorems, the sense, and propriety of the two definitions of perpendicular, and parallel planes, appear manifest.

THEOREM VIII.

Right-lines (AB, CD) parallel to one and the same right-line (EF) though not in the same plane with it, are also parallel to each other.

Let GH and GI be drawn perpendicular to EF, in the planes AF and ED of the proposed parallels. Thenⁿ shall GF be perpendicular to the plane passing by HGI; and HB, ID will also be perpendicular to the same plane^o, and therefore parallel to each otherⁿ.

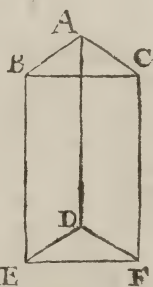


^o 5. 7.
ⁿ 4. 7.

THEOREM IX.

If two right-lines (AB, AC) meeting each other, be respectively parallel to two other right-lines (DE, DF) also meeting each other, and not being in the same plane with them; the angles (BAC, EDF) contained by those lines, will be equal.

Take AB, AC, DE, DF all equal to each other, and let BE, AD, CF, BC, EF be drawn. Then AB and ED, as well as AC and DF, being equal and parallel^r, BE and CF will be both equal, and parallel, to AD^r, and therefore equal, and parallel, to each other^s; whence BC is also equal to EF^r; and so, the triangles ABC, DEF being mutually equilateral, the angles BAC, EDF are likewise equal^t.



^r Hyp. and Constr.
^r 26. 1.
^s Ax. 1. and 8. 7.

^t 14. 1.

THEOREM X.

If two right-lines (AB, AC) meeting each other, be respectively parallel to two other right-lines

(DE, DF) also meeting each other, and not being in the same plane with them, the planes (BAC, EDF) extended by those lines, will be parallels.

^u 8. 7.

^w Constr.

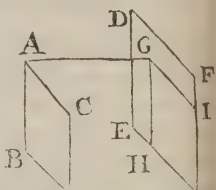
^x 5. 1.

^a 2. 7.

^b Constr.

^c 7. 7.

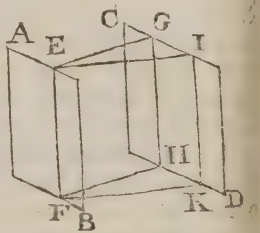
Let AG be perpendicular to the plane BAC, meeting the plane EDF in G; in which last plane, let GH and GI be drawn parallel to ED and DF; and they will also be parallel to AB and AC^u; whence, seeing the angles GAB and GAC are both right^w, AGH and AGI must likewise be right-angles^x; and so AG being perpendicular to the plane EDF^a (as well as to BAC^b), the two planes are parallel to each other^c.



THEOREM XI.

The sections (EF, GH) made by a plane (EFHG) cutting two parallel planes (AB, CD,) are also parallel, the one to the other.

Let EG and FH be drawn parallel to each other, in the plane EFHG; also let EI, FK be perpendicular to the plane CD, and let IG, KH be joined: Then, EG being parallel to FH^d, and EI to FK^e, the angle GEI is = HFK^f; but the angle EIG



^d Constr.

^e 4. 7.

^f 9. 7.

^g Def. 1. 7.

^h Def. 3. 7.

ⁱ 15. 1.

^k 26. 1.

is also = FKH, being both right-angles^g; and EI is = FK^h: Therefore EG will be equalⁱ (as well as parallel) to FH; and consequently EF likewise parallel to GH^k.

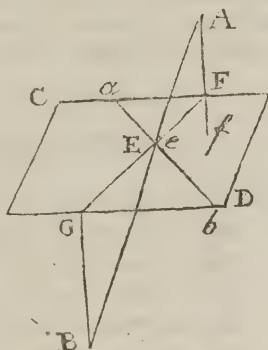
COROLLARY.

It appears from hence, that parallel lines, terminated by the same parallel planes, are equal to each other.

THEOREM XII.

If, from the two extremes of a right-line (AB) cutting a plane (CD,) two perpendiculars (AF, BG) be drawn to the plane; the right-line (FG) joining the points where they meet the plane, will pass through the point (E) in which the proposed line (AB) cuts the plane, so as to be divided by it into two parts (FE, EG,) having the same ratio to each other as those two perpendiculars (AF, BG.)

For, if AF be produced to *f*, the lines Af and BG (which are both perpendicular to the plane CD ^l) will be parallel to each other ^m; therefore AB and FG being both in the same plane with these parallels, in which their extremes are positedⁿ, they must necessarily (as they are not themselves parallels) intersect each other; And so the alternate angles FAE, GBE being equal^o, as well as the opposite ones FEA, GEB^p, thence will ^p3. 1. $FE : EG :: AF : BG$ ^q; which was to be demon-^r14. 4. strated.



^l Hyp.
^m 4. 7.

ⁿ Def. 6. 1.

^o 7. 1.

^p 3. 1.

^q 14. 4.

COROLLARY.

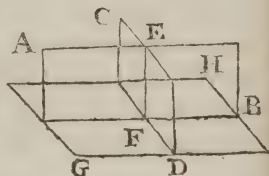
Hence, if in the plane CD, the lines FC, GD be made parallel, the one to the other, and in them be taken $Fa = FA$, and $Gb = GB$; then will the line (ab) joining the points a and b, cut FG in the very same point in which it is cut by AB. For, if e be taken as the intersection of ab and FG, the triangles aFe, Geb will be equiangular^r; whence $Fe : eG :: Fa$ (FA) : ^sGb (GB) : : $FE : EG$ ^t. Therefore, seeing FG is divided in one and the same ratio, both by e and E, these points must necessarily coincide^u.

^r 3. and
^s 7. of 1.
^t 14. 4.
^u 12. 7.
^v Ax. 2.
and 5.
4.

THEOREM XIII.

If two planes (AB, CD) cutting each other, be both perpendicular to a third plane (GH,) their common section will also be perpendicular to the same plane (GH.)

For, from the extreme point F of the common section, let the right-line FE be erected perpendicular to the plane GH: which line being in both the planes AB, CD^l, it must necessarily be their common



^lCor. 2. to 6. 7.

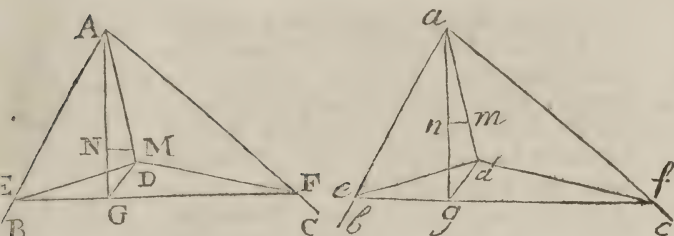
^mConstr.

Therefore the common section is perpendicular to the plane GH^m.

THEOREM XIV.

If, from the angular points (A, a) of two equal angles (BAC, bac) two right-lines (AD, ad) be drawn, or elevated on high, above the planes of the said angles, so as to form equal angles with the lines first given, each to its correspondent ($\angle DAB = \angle dab$, $\angle DAC = \angle dac$,) and if, from any points (M m) in those elevated lines perpendiculars (MN, mn) be let fall upon the planes (BAC, bac) of the first-mentioned angles; these perpendiculars will be, in proportion, as the parts (AM, am) of the elevated lines included between them and the angular points (A, a) first named.

Make AD and ad equal to each other; and in the planes ADB, ADC, adb, adc, draw DE, DF, de, df perpendicular to AD and ad; and from their intersections with AB, AC, ab, ac, draw EF and ef, meeting AN and an (produced) in G and g, and let D, G, and d, g, be joined.



The angles ADE, ADF being both right-onesⁿ, notⁿ Constr.
 only the line AD^o, but the plane ADG extended by^o 2. 7.
 , is perpendicular to the plane EDF^p. But the same^p 6. 7.
 plane ADG is also perpendicular to the plane EAF^p:
 Therefore the common section EF is likewise perpen-
 dicular to the plane ADG^q; and consequently the^q 13. 7.
 angle EGA a right-one^r. By the very same argu-^r Def. 1. 7.
 ment, *ega* is a right-angle. Now the triangles ADE,
de; ADF, *adf* being equal in all respects^s, and the^s Hyp. and
 angle EAF=*caf*^t, the triangles AEF, *aef* are also^t 15. 1.
 equal and alike^u; and so, the angle AEG being=^u Hyp.
ega, and AE=*ae*, thence is AG=*ag*^w,^w Ax. 10. 1.
 and the angle DAG (MAN)=^x *dag* (*man*), because^x 16. 1.
 DG, *adg* are both right-angles^o. Therefore MN:
mn :: AM : *am*^y.^y 14. 4.

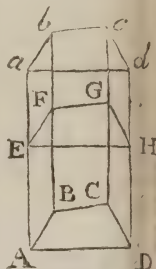
COROLLARY.

Hence, the two perpendiculars MN, *mn* subtend equal
 angles at the points (A, a) from whence the two
 elevated (or inclining) lines are drawn.

THEOREM XV.

If any solid (AC) having a rectilinear base
 (ABCD,) whereof the planes (Ab, BC, CD, AD)
 of the sides are parallelograms, be cut by a
 plane parallel to the base, the section (EFGH)
 will be equal, and similar to the base.

For, the plane EFGH being parallel to ABCD^z, EF is therefore parallel to AB^a; and so, AF being a parallelogram^b, EF is equal (as well as parallel) to AB^c. In the same manner is FG equal, and parallel to FG, &c. Whence also the angle EFG is = the angle ABC^d; and so of the rest. Therefore EFGH is both equilateral and equiangular to ABCD.



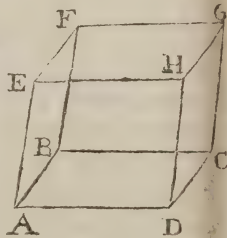
COROLLARY.

Hence, *the opposite bases of a prism are equal and similar (as well as parallel) to each other.*

THEOREM XVI.

If, from one of the angular points (A) of any parallelogram (AC) a right-line (AE) be elevated above the plane of the parallelogram, so as to make any angles (EAB, EAD) with the two contiguous sides (AB, AD) and there be also drawn, from the three remaining angular points, three other right-lines (BF, CG, DH) parallel, and equal to the former (AE;) then, the extremes of those lines being joined, I say, the figure (AG) thus described, will be a parallelepipedon.

For AE, BF, CG, DH being all parallel to each other^e, AE and BF are in the same plane^f, as are also AE and DH, &c. Therefore, all these lines being^g equal among themselves, AF, AH, DG, and BG are parallelograms^h; and so, EF being parallel to AB, parallel to DC, parallel to HGⁱ, EF and HG are in the same plane^j; and EG is also a parallelogram^k, equilateral to its op-

^e Hyp. and

8. 7.

^f Def. 13.

1.

^g Hyp.^h 26. 1.ⁱ 8. 7.^k 25. 1.

posite AC; but EG is equiangular, and parallel (as well as equilateral) to its opposite AC; because, EF being parallel to AB, and EH to AD, the angle FEH is therefore = BAD^l, and the plane EG parallel^{9.7.} to the plane AC^m. And in the same manner, the^{m 10.7.} other opposite parallelograms appear to be equiangular and parallel (as well as equilateral.) Therefore the solid AG, bounded by them, is a parallelepipedonⁿ. ^{n Def. 7. of 7.}

COROLLARY.

If the angle A of the parallelogram AC be a right-one, and AE be erected perpendicular to the plane AC; then will the parallelepipedon be a rectangular one: for, all the three contiguous planes AC, AF, AH being rectangular^o, their opposites will be^{o Hyp. and 24. 1.} rectangular likewise^p; and so, the angles HGF, HGC being right-ones, HG will be perpendicular^{p 16.7.} to the plane GB^q; and consequently both the planes^{q 2.7.} EG and DG likewise perpendicular to the plane BG^r. And so of the rest. ^{r 6.7.}

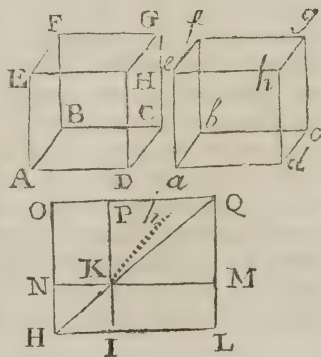
SCHOLIUM.

In this Theorem, a way to describe a parallelepipedon of any given dimensions, is indicated; and the consistence of the 7th and 9th definitions evinced.

THEOREM XVII.

Rectangular parallelepipedons (AG, ag) standing upon equal bases (AC, ac), and having equal altitudes (AE, ae) are equal.

Let the rectangles OK and KL, equal and like to the bases AC and ac of the two solids, be so formed, that NK may be in the same straight line with KM; then shall PK be also in the same straight line with KI^s; and the figures NI, PM, OL, formed by producing the sides of the two rectangles, will likewise be rectangles^t,



^{t 2. 1.}

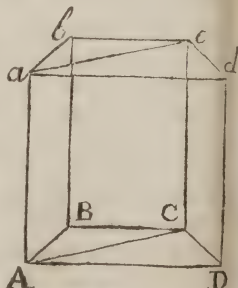
^{t Cor. to 24. 1.}

- Now, HK, and QK being drawn, the triangle NHK
 " 24. 1. =IHK^u, PKQ=MKQ^u, and the rectangle OK=
 " Hyp. LK^w; and consequently (by the addition of equals
 OQKH=LQKH. Therefore, HKQ being a diagona
 " 24. 1. to the rectangle OL^x, dividing it into two equal, and
 and Ax. 2. like triangles OQH, LQH^u, if upon these, as bases
 two upright prisms be conceived to be erected, of the
 same common altitude (Kh) with the proposed solids,
 " Ax. 1. 7. these prisms will also be equal^y. But the former of
 " Cor. to these is composed^z of three prisms, on the bases
 18. 7. OPKN, NHK, KPQ; and the latter of three others
 on KMLI, HIK, KMQ; the second and third o
 which, in both ranks, are respectively equal^y. There-
 fore the remaining two, on the bases OPKN, KMLI
 " Ax. 5. must also be^r equal. But the former of these is=
 AG^y, and the latter=ag: Therefore, also, is AG=
 " Ax. 1. ag^a.

THEOREM XVIII.

If, at the angular points of any given right-lined figure (ABCD), equal perpendiculars (Aa, Bb, Cc, Dd) be erected, and the extremes of these (a, b; b, c, &c.) be joined; an upright prism (AabcdDCBA) on the given base (ABCD) will thereby be formed.

- For, Aa, Bb, Cc, Dd being
 " Constr. all equal^b, and parallel^c, it is
 " 4. 7. evident that AabB, BbcC, &c.
 " 26. 1. are parallelograms^d; and that
 the planes of these are all per-
 pendicular to that of the base
 " 6. 7. ABCD^e (since Aa, Bb, Cc are
 so, by construction). Moreover
 it will appear, that abcd is one
 plane figure, parallel to ABCD; **A** **D**
 for, the lines ac, AC (when a, c, and A, C are
 " 26. 1. joined) being parallels^f (as well as ab, AB; ad, AD),



the plane abc is, therefore, parallel to $ABCD$ (or ϵ 10. 7. BCD); and acd is likewise parallel to $ABCD$.
 But abc , and acd are in one plane; Because Aa being
 perpendicular to both of them h , and i consequently to h Cor to
 all the lines ab , ac , ad ; these must necessarily be all $7. 7.$
 in one k plane, parallel to $ABCD$; which was to be i Def. 1. 7.
 demonstrated 1 . k Cor. to
 $2. 7.$
 1 Def. 6.

COROLLARY.

It appears from hence, that, if upon all the parts
 ABC , ACD , into which any rectilinear figure
 $ABCD$ is divided, upright prisms ($AabcCB$,
 $AacdDC$) of the same altitude be constituted; these
 prisms will form one prism, on the (whole) given
 base $ABCD$; Because abc and acd form one con-
 tinued plane superficies $abcd$ m , parallel to $ABCD$. n 18. 7.

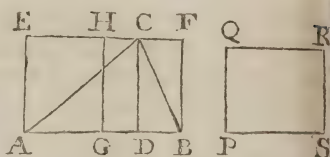
SCHOLIUM.

After the same way, a prism, any how inclining
 on the given base $ABCD$, may be constructed; by
 giving to Aa the proposed inclination, and then
 drawing Bb , Cc , Dd parallel, and equal thereto.
 For $AabB$, $BbcC$, &c. will (still) be parallelograms f : f 26. 1.
 And, that $abcd$ is one plane, parallel to $ABCD$, will
 also appear (in the same manner); if a perpendicular
 from a to the plane $ABCD$, be conceived to be
 drawn.

THEOREM XIX.

If, on equal bases (ABC , $PQRS$), an upright tri-
 angular prism, and a rectangular parallepipe-
 don be erected, of the same altitude; the two
 solids, themselves, will be equal.

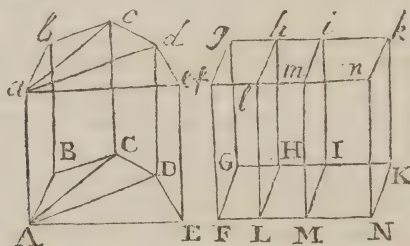
Let CD be perpendicular to AB, and let the rectangles ADCE, BDCF be completed; also let GH be drawn parallel to AE, bisecting AB in G; so shall $AGHE = \frac{1}{2}ABFE^b = ACB^c = PQRS^d$. Now, the two prisms on ADC and BDC into which, *that* on ABC may be divided^e, will be respectively equal to two others on the equal and similar^f bases, AEC and BFC^g: and consequently the prism on ACB = half the prism on AEFB^h = half the two prisms on AGHE and BGHF = the prism on AGHE = the prism on^k PQRS*.



^b 1. 2.
^c Cor. to 2. 2.
^d Hyp.
^e Cor. to 18. 7.
^f Ax. 24. 1.
^g Ax. 1. 7.
^h Ax. 4. 1.
ⁱ Ax. 3. 1.
^k 17. 7.

THEOREM XX.

Every upright prism ($\Delta ace\Lambda$) is equal to a rectangular parallelepipedon (Fk) of equal base and altitude.



Let AC, ac, AD, ad be drawn; and in the base FK of the parallelepipedon, let HL, IM, be drawn parallel to FG, in such sort, that the rectangles FGHM, IN may be respectively equal to the triangles ABC, ACD, ADE^k: Then also shall the prism

^k 6. 6.

* In this Theorem, the representations of the prisms are not described; because, a multiplicity of lines, tends to produce confusion in the mind of a learner; especially where solids are represented. The schemes, however, may be formed, at large, by those who think proper to do it: But every tittle of the demonstration will remain the same.

(*AabcCBA*) on *ABC*, be equal to the parallelepipedon (*Fh*) on *FH*¹; and the prism (*AacdDCA*) on *ACD*,^{19.7.} equal to the parallelepipedon (*Li*) on *LI*¹; and consequently the whole prism (*AaceA*) on *ABCDE*, equal to all the parallelepipedons on *FK*, which form the parallelepipedon (*Fk*); because *Ll*, *Mm* are in the plane *Fn*^m, and *Hh*, *Ii* in the plane *Gk*^m.

^m 18. 7.
and Cor.
2. to 6. 7.

COROLLARY.

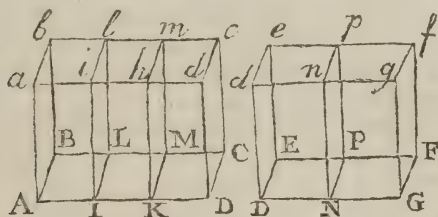
Hence all upright prisms having equal bases, and altitudes, are equal among themselves.

SCHOLIUM.

In the very same manner, the aggregate of any number of prisms, of one common altitude, will appear to be equal to one single prism, or parallelepipedon, of the same altitude, whose base is equal to the sum of all theirs.

THEOREM XXI.

Rectangular parallelepipedons (*Ac*, *Df*) having equal altitudes (*Aa*, *Dd*) are in the same proportion as their bases (*AC*, *DF*).

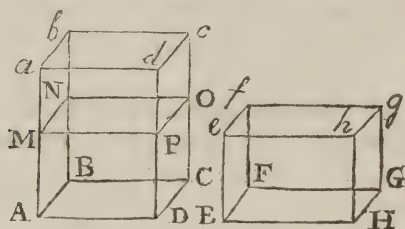


Let the proportion of the base *AC* to the base, *DF*, be that of any one number *m* (3) to any other number *n* (2). Let *AC* be divided into *m* (3) equal parts, (or rectangles) *AL*, *IM*, *KC* (by dividing *AD* into that number of equal parts^m, and drawing *IL*, *KM* parallel to *AB*)ⁿ: And let *DF* be divided, in like manner, into *n* (2) equal parts, or rectangles, *DP*,^{11.5.} ^{9.5. and 1.2.}

NF: Which parts, taken singly, will be equal in magnitude, to those of the former division^o: and so
^o Hyp. and Ax. 8. 4. the parallelepipedons upon them (Al, Im, Kc, Dp, Nf)
^p 17. 7. will likewise be all equalⁿ: Therefore the solid Ac is in proportion to the solid Df, as the number of parts in Ac to the number of equal parts in Df^q, or as the number of parts in AE to the number of equal parts in DF, that is, as AC to DF^q.—If the bases are supposed to be *incommensurable*, the solids will still be in the same ratio with them, as appears from the reasoning laid down in the *Scholium* to *Theorem VII.* Book IV.; which is equally applicable in this case.

THEOREM XXII.

Rectangular parallelepipedons (Ac, Eg) standing upon equal bases (Ac, Eg) have the same ratio as their altitudes (Aa, Ee).



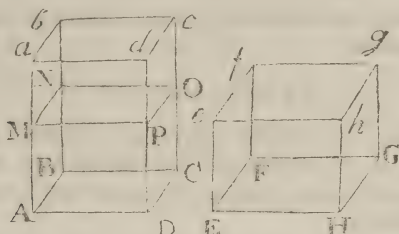
Let AO be a parallelepipedon on the base AC, whereof the altitude AM is equal to that (Ee) of the parallelepipedon Eg: So shall the solid AO = the solid Eg^s. (But if Ab and AN be considered as bases)
^s 17. 7. it will be Ac : AO (or Eg) :: Ab : AN^t :: Aa : AMⁿ
^t 21. 7. (or E) : which was to be proved.
ⁿ 7. 4.

COROLLARY.

Hence, and from the preceding Theorem, it follows, that all upright prisms are, as the bases, when the altitudes are equal; and as the altitudes, when the bases are equal; all such solids being (by Theorem XX.) equal to rectangular parallelepipedons of equal base and altitude.

THEOREM XXIII.

Upright prisms and parallelepipeds (Δc , Eg) which have their bases and altitudes reciprocally proportional ($AC : EG :: Ee : Aa$), are equal to each other.



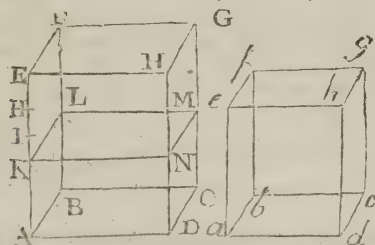
Let AO be a prism on the base AC, whereof the altitude AM is equal to that (Ee) of the prism Eg .

Then $AO : Eg :: AC : EG^w :: Ee (AM) : ^x Aa :: ^w 21.7.$

$O : ^y Ac$; and consequently $Eg = Ac^z$.
 ^{x} Hyp.
 ^{y} 22.7.
 ^{z} Ax. 4.
of 4.

THEOREM XXIV.

Similar upright prisms and parallelepipeds (ΔG , ag) are, to one another, in the triplicate ratio of their altitudes (ΔE , ae).



Having made $AH = ae$, take AE, AH, AI, AK in continued proportion^a; and let AM be a prism on^a 13.5. the base AC , whereof the altitude is AK . Then (because of the similar planes) it will be $AC : ac :: ^b 26.4.$ $B^2 : ab^2 :: ^c AE^2 : ae^2 (AH^2) :: AH^2 : ^e AI^2 :: AH^c$ Cor. to (e) : AK^e ; and so, the bases and altitudes of the^{11.4.} ^{27.4.}

23. 7.
 Ax. 1.
 of 4.
 22. 7.
 Def. 7.
 of 4.

solids AM, *ag* being reciprocally proportional, the solids themselves are equal^f; and therefore, $AG : ag :: AG : AM^g :: AE : AK^h$. But the ratio of *AE* to *AK* is triplicate to that of *AE* to ⁱ*AH* (or *ae*). Therefore, &c.

COROLLARY I.

Hence, *cubes are in the triplicate ratio of their sides, or altitudes.*

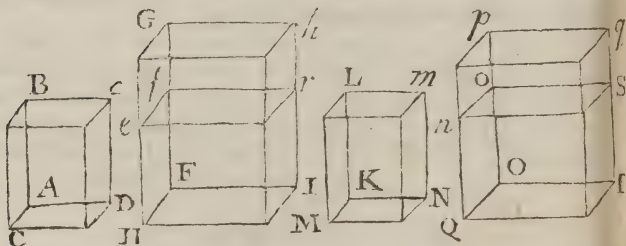
COROLLARY II.

Hence, also, *all similar upright prisms, are to one another, as the cubes of their altitudes; since both prisms and cubes, are in the same triplicate ratio of the altitudes.*

THEOREM XXV.

Rectangular parallelepipeds, contained under the corresponding lines of three ranks of proportionals, are themselves proportionals.

I say, if $\begin{cases} AB : FG :: KL : OP, \\ AC : HF :: MK : QO, \\ AD : FI :: KN : OR, \end{cases}$



then shall the solid (cc) contained under the three first antecedents, be to that (hh) contained under their three consequents; as the solid (mm) contained under the three other antecedents, is that (nn) contained under the three remaining consequents.

Let Hr and Qs be parallelepipeds on the bases HI^* and QR , of the same altitude with Cc and Mm respectively.

Then shall $Cc : Hr :: \text{base } CD : \text{base } HI^k$; $^k 21. 7.$

And $Mm : Qs :: \text{base } MN : \text{base } QR^k$.

But the four bases, because of the proportionality of their sides, are themselves proportionals^l: and so, by ^l 11. 4. equality, the ratio of Cc to Hr , is the same as the ratio of Mm to Qs . And the ratio of Hr to Hh is likewise the same as that of Qs to Qq (because $Hr : Hh :: Ff' (AB) : FG^m :: KL (Oo : Op^n :: Qs : ^m 22. 7. Qq^m$.) Therefore shall the ratio of Cc to Hh , be also ⁿ Hyp. the same, as the ratio of Mm to Qq^o . $^o 5. 4.$

COROLLARY.

Hence *the cubes of four proportional lines are proportional.*

THE END OF THE SEVENTH BOOK.

ELEMENTS
OF
GEOMETRY.

BOOK VIII.

POSTULATES.

1. **T**HAT, of any two unequal magnitudes, of the same kind, the less may be multiplied so often, till it exceed the greater.

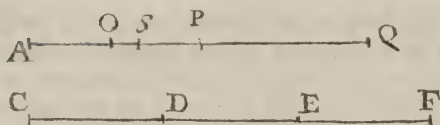
2. That, a right-line may be taken so small, that its square shall be less than any superficies assigned.

3. That, the circumference of a circle is greater than the perimeter (or the sum of all the sides) of any inscribed polygon; and less than the perimeter of any polygon described about the circle.

What is required to be granted, in the second of these three Postulates, might be effected and proved, in form, by means of the First; but being itself more obvious (if possible) than even that, it seemed unnecessary to make it depend upon it.

LEMMA 1.

If from the greater (ΔQ) of two unequal magnitudes (ΔQ , CD) there be taken the half (PQ), and from the remainder (ΔP) be again taken the half (PO), and so on, continually; there shall at length be left a magnitude, less than the least (CD) of the two magnitudes first propounded.



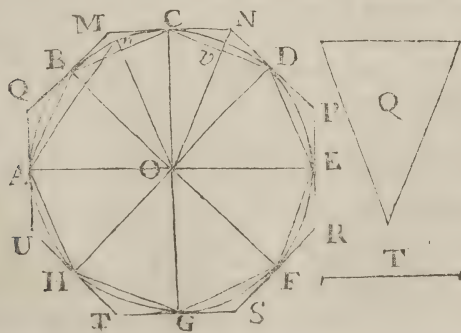
Take $DE = CD$, $EF = CD$, and let this be so often done, till the multiple CF exceed AQ^a . Let the proposed bisections of AQ , AP , &c. be continued, till the parts PQ , OP , AO be equal in number to the parts EF , DE , CD . Now $AP (\frac{1}{2}AQ^b) \sqsupset \frac{1}{2}CF^b \sqsupset CE$. ^{b Hyp.} And, in the same manner, $AO (\frac{1}{2}AP) \sqsupset \frac{1}{2}CE (CD;)$ which was to be done.

SCHOLIUM.

When the magnitudes given (AQ , CD) are right-lines, a part, or measure (AS) of the one, less than the other, may be found at one operation^c; by taking ^{c 11. 5.} AS the same part of AQ , as CD is of CF^c . For, the whole AQ being less than the whole CF^b , the part AS will also be less than the part CD^d . ^{d Cor. 1. 4.}

THEOREM 1.

Two polygons may be formed, the one in, the other about a given circle, which shall differ less from each other (and consequently from the circle itself) than by any assigned magnitude (Q) however small.

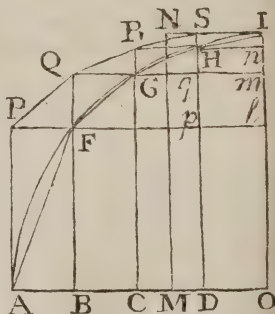


Let T be the side of a square equal to, or less than Q^e ; and in the circle apply $An = T^f$: and, having drawn the two perpendicular diameters AE , CG , proceed by a continual bisection of the angles at the center, till you arrive at an angle AOB less than the angle AOn subtended by An^g : Inscribe the regular^h polygon $ABCDEFGH$, by making the angles BOC , COD , &c. equal to AOB : and let another regular polygon $QMNPRSTU$, of the same number of sidesⁱ, be described about the circle; which will exceed the inscribed one by a magnitude less than Q .

For, if to any angle N of the greater, ON be drawn, it will bisect the same^k, and will cut the side CD of the inscribed polygon at right-angles^l (in v): And so, the triangles OCN , vCN being equiangular^m, they (and consequently their doubles $OCND$, CND) will be in proportion to each other, asⁿ OC^2 to Cv^2 , or as^o AE^2 to AB^2 . And it is manifest, that the whole circumscribing polygon ($OCND + ODPE$, &c.) must be to its whole excess ($CND + DPE$, &c.) above the inscribed one, in the same^o proportion of AE^2 to AB^2 . But the first *antecedent* is less than the *second*, or than a square described about the circle^p: Therefore the first *consequent* ($CND + DPE$, &c.) is also $\sqsubset^q AB^2 \sqsubset^r An^2 (T^2) \sqsubset^s Q$.

Otherwise.

Let AIO be one quadrant of the proposed circle; and on the radius OI , make^t the rectangle $OMNI = \frac{1}{4} T^2$, (T being as before:) Take OD a part of OA , less than OM^u ; and, having made AB , BC , &c. each $= OD$, draw AP , BFQ , CGR , DHS perpendicular^w to AO , meeting the circumference in A , F , G , H ; through which points, parallel to AO^x , draw PfL , QgM , Rhn meeting AP , BFQ , CGR , in P , Q , R : Join PQ , QR , RS , SI , as also AF , FG , GH , HI . Then will the two polygons $OAFGHI$



and OAPQRSI (one of which is less, and the other greater than the quadrant) differ less from each other, than by $\frac{1}{2}$ of the proposed quantity Q.

For, that the former OAFGHI is less than the quadrant, in which it is inscribed, is manifest^y: And, that^y Ax. 2. the latter is greater than the quadrant, will also plainly appear; seeing two sides AP, SI, only, touch the circumference^z; all the rest, PQ, QR, RS, falling wholly⁶ 3. above it, as being sides of triangles PFQ, QGR, RHS, formed out of the circle^a. Now the excess of the po-^a Constr. lygon OAPQRSI above OAFGHI, is composed of^{and Ax. 2.} the triangle PAF ($=\frac{1}{2}ODpl$) and of all the parallelograms PFGQ, QGHR^b, RHIS (for they are such^c, ^b Ax. 3. because PF (AB,) QG (BC) are equal, as well as^c 26. 1. parallel^d.) Therefore PFGQ being $=plmq^e$, QGHR = ^d Constr. $qmnH^e$, &c. the said excess will consequently be = ^{and 26. 1.} $\frac{1}{2}ODpl + lpSI^b \supset ODSI^f \supset OMNI (\frac{1}{2} T^2) \supset \frac{1}{2} Q^g$; ^e Cor. 2. to 2. 2. which was to be done.—This last construction is^f Ax. 2. equally applicable to other curvilinear figures; the^g Hyp. former is peculiar to the circle.

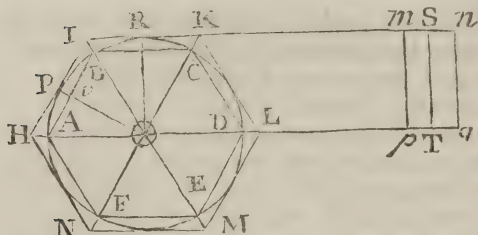
COROLLARY.

It follows from hence, that *a magnitude, which is greater than any polygon that can be described in, and less than any polygon that can be formed about, a given circle, must be equal to the circle itself*: seeing that a polygon may be inscribed, which (as well as that formed about the circle) shall exceed any quantity less than the circle itself, be the difference ever so small; and because a polygon may be formed about the circle, which (as well as that in the circle) shall be less than any quantity that exceeds the circle.

THEOREM II.

Every circle (ACE) is equal to a rectangle (ORSI) under its radius (OR) and a right-line (OT) equal to half the circumference.

It is evident, in the first place, that the proposed rectangle ORST is greater than any polygon ABCDEF



that can be described in the circle: For, drawing OA, OB, &c. And also Ov perpendicular to AB; it is plain, that the triangle AOB (^h $Ov \times \frac{1}{2}AB$) will be less than ⁱ $OA \times \frac{1}{2}AB$ (or $OR \times \frac{1}{2}AB$;) And, in the same manner, $BOC \sqsubset OR \times \frac{1}{2}BC$, &c. Consequently, the whole polygon ABCDEF is less than ^k $OR \times \frac{1}{2}AB + OR \times \frac{1}{2}BC$, &c. that is ^l, less than a rectangle (Om under OR and Op=half the perimeter (AB+BC+CD, &c.) But this rectangle (Om) is, itself, less than OS, because, Op (half the perimeter of the polygon is less than OT ^m (half the circumference of the circle. Consequently the polygon ABCDEF is less than the rectangle OS.

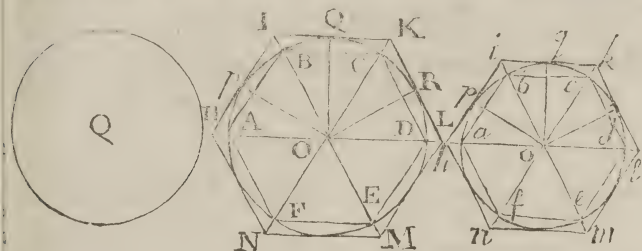
But, secondly, it will appear, that the same rectangle ORST is less than any polygon HIKLMN that can be described about the circle: For, if OH, OI, &c. be joined and the radius OP be drawn to the point where HI touches the circle; then will the triangle HOI=ⁿ $OP \times \frac{1}{2}HI$ ($=OR \times \frac{1}{2}HI$.) In the very same manner $IOK = OR \times \frac{1}{2}IK$, &c. and therefore the whole polygon HIKLMN=^o $OR \times \frac{1}{2}HI + OR \times \frac{1}{2}IK$, &c.=^p a rectangle (On) under OR and Oq=half the perimeter (HI+IK+KL, &c.;) which rectangle is manifestly greater than OS, since Oq (=half the perimeter of the polygon) is greater than OT ^q.

Seeing, therefore, that the rectangle OS is greater than any polygon that can be described in the circle, and less than any polygon that can be described about the circle; it must be equal to the circle ^r.

THEOREM III.

All circles (ACE, ace) are in proportion to one another, as the squares of their radii ($AO^2 ao^2$).

Let $Q : \text{circle } ace :: AO^2 : ao^2$; then I say, that $Q = \text{circle } ACE$. For, first it is evident that Q is greater than any polygon $ABCDEF$ that can be des-



cribed in the circle ACE : Because, if another polygon $abcdef$, similar thereto, be described in the circle ace ; then will polyg. $ABCDEF : \text{polyg. } abcdef :: AO^2 : ao^2 :: Q : \text{circle } ace$; where the first consequent (polyg. $abcdef$) being less than the second (or, than the circle in which it is inscribed^w) it is manifest, that the first antecedent $ABCDEF$ must also be less than the second Q .

In the same manner it will appear, that Q is less than any polygon $HIKLMN$ that can possibly be described about the circle ACE : For, if about the other circle ace , a similar polygon $hiklmn$ be described; then will $HIKLMN : hiklmn :: AO^2 : ao^2 :: Q : \text{circle } ace$; where the first consequent ($hiklmn$) being greater than the second (ace),^b the first antecedent $HIKLMN$ must therefore be also greater than the second Q .

Therefore, seeing that Q is greater than any polygon that can be described in the circle ACE , and less than any polygon that can be described about the circle; it must be equal to the circle^d.

^d Cor. to 1.
8.

SCHOLIUM.

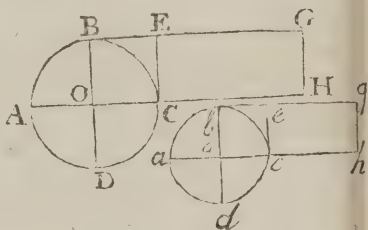
After the same manner, other similar curvilinear figures are proved to be in proportion to one another, as the squares of their diameters, or other homologous dimensions; by means of the second construction of the first proposition; it being very easy to demonstrate that the polygons formed from thence, whether both

within, or both without two similar figures, will themselves be similar.

THEOREM IV.

The circumferences of all circles (ABCD, abcd) are in the same proportion as their radii (OB, ob.)

Let OE, oe be squares on the radii OB, ob; and let OG, og be two rectangles contained under the same radii and right-lines OH, oh, respectively equal to the semi-circumferences



ABC, abc. Then, these rectangles being equal to the circles themselves^d, it will therefore be, OE : OG :: oe, og^e. And in this same ratio are^f also the bases OC, OH; oc, oh: whence (by equality and alternation) OC (OB) : oc (ob) :: OH : oh :: 2OH (circumf. ABCD^g) : 2oh (circumf. abcd.)

^d 2. 8.

^e 3. 8.

^f 7. 4.

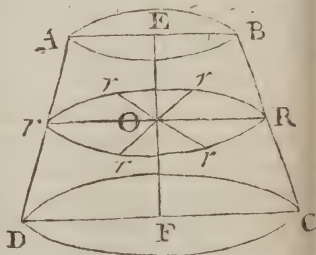
^g Hyp.

LEMMA 2.

If a solid (AC) generated by the revolution of any plane figure (EBCF) about a quiescent axis (EF,) be cut by a plane perpendicular to the axis; the section will be a circle, having its center in the point (o) where it meets the axis.

For, from O, in the generating plane EBCF, draw OR perpendicular to the axis EF, meeting BC in R.

Then, since this line OR, during the whole revolution, every where preserves its perpendicularity to the axis EF, it is there-



are always in the plane passing through O perpendicular to the said axis^k: and consequently, as its^h Cor. to 2 length also continues the same in every position, 7. the line Rrrrr described, in that plane, by the extreme point R, by which the section is bounded, must be the circumference of a circleⁱ, of which the point O is the center. Def. 33. 1

COROLLARY.

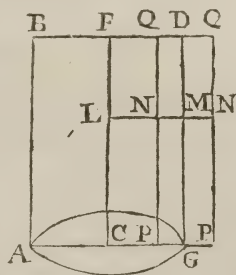
Hence, the bases of cylinders and cones, and all sections parallel to them, are circles.

LEMMA 3.

A right-line (PQ) standing perpendicular to the plane of a cylinder's base (and not exceeding the axis CF) falls wholly within, or wholly without the cylinder, according as the point (P) on which it insists, is situated within, or without the circumference of the base.

From the center C, to the given point P, draw CP; take, in CF and PQ, any two equal distances CL, PN, and let LN be drawn, meeting the surface of the cylinder M.

Because CL and PN are parallel^k, and therefore both in the same planeⁱ, LN is parallel, and equal to CP^m. Therefore, when CP is less than the radius CG, LN will be less than CG, or than its equal LMⁿ; and so the point N must fall within the cylinder^o. And the same is equally true with regard to any other point in the line PQ. But, when CP is greater than CG, LN will also be greater than HG (LM;) and the point N will then fall out of the cylinder^o.



^k 4. 7.

ⁱ Def. 13.

^m 26. 1.

ⁿ 24. 1.

^o Def. 12.

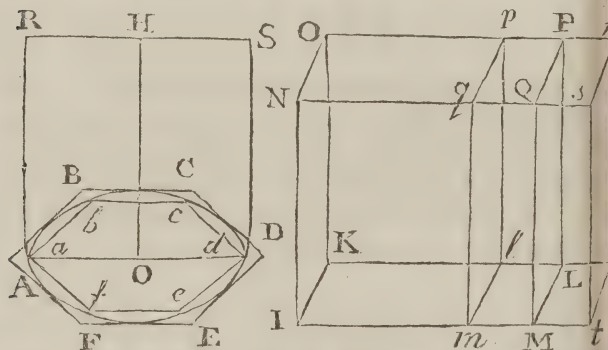
^o Ax. 2. 1.

THEOREM V.

Every cylinder is equal to a rectangular parallelepipedon of equal base and altitude.

I say, if the base ace of the cylinder aS be equal to the base $IKLM$ of the rectangular parallelepipedon IP , and the altitude OH of the former be also equal to the altitude KO of the latter: then the two solids will be equal.

For, first, it is evident, that the cylinder exceeds any parallelepipedon (Ip ,) of the same given altitude, whose base $iklm$ is less than the base (ace) of the cylinder.



linder. Because a polygon ($abcdef$) may be described in the circle ace , that shall exceed $IKlm$ ^p; upon which, an upright prism (of the given altitude) may be constituted^q; which will be less than the cylinder, as being wholly contained therein; since (*by Lemma 3.*) all right-lines drawn perpendicular to the base, to the planes of the sides^r, from any points in ab, bc, cd , fall wholly within the cylinder, and consequently to the planes themselves, in which they are. But this contained prism is greater than the parallelepipedon Ip : therefore the cylinder itself must, necessarily, be greater than Ip ^s.

^p 1. 8.
^q 18. 7.
^r Cor. 2. to 6. 7.
^s 21. 7.
^t Ax. 2.

In like manner it will appear, that the cylinder is less than any parallelepipedon, Ir (of the same altitude) whose base $IKvt$ exceeds that of the cylinder.

a polygon (ABCDEF) may be described about
 circle *ace* that shall be less than $IKvt^u$; upon $^u 1. 8.$
 which a prism may be constituted ^{u} , which, though $^u 18. 7.$
 e. than Ir^z , will, nevertheless, exceed the cylinder ^{u} . $^z 21. 7.$
 Therefore, seeing that the cylinder can neither be $^u Ax. 2.$
 e. nor greater than IP ; it must necessarily be equal
 to it.

COROLLARY.

Hence, whatever is demonstrated (in the 21st, 22nd,
 and 23rd Theorems of the preceding book,) with
 respect to the proportions of prisms, holds equally
 true in cylinders also; being equal to prisms of
 equal base and altitude ^{z} .

$^z 20. 7.$
 and $5. 8.$

SCHOLIUM.

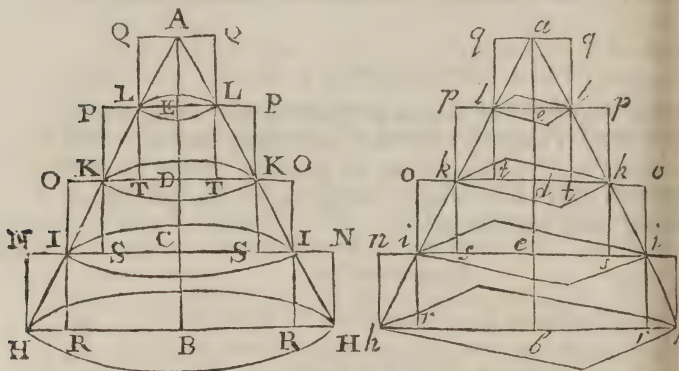
From the same demonstration, it will likewise ap-
 pear, that every regular solid, whose sections, by
 planes perpendicular to the base, are all rectangles,
 equal to a parallelepipedon of equal base and alti-
 tude; and consequently, that all solids of this kind
 which may be comprehended under the name of
 (cylindroids) will be equal among themselves, when
 their altitudes, as well as bases, are equal.

LEMMA 4.

Let two solids (HAH, hah) of the same altitude,
 have their sections by planes parallel to the
 bases, at all equal distances therefrom, equal
 to each other; it is proposed to demonstrate
 (under certain restrictions specified hereafter)
 that the solids themselves will be equal.

Let II, KK, &c. ii , kk , &c. be sections of the two
 solids by planes parallel to the bases HH, hh , dividing
 the altitudes AB, ab into parts BC, CD, &c. bc , ed ,
 &c. all mutually equal to each other. Then, every
 to corresponding sections being equal ^{m} ($HH = hh$, ^{m} Hyp.
 $II = ii$, &c.) the upright solids HNNH, $hnnh$; IOOI,

20. 7. and Sch. 5. 8. *iooi*, &c. formed thereon, will also be respectively equal one to anotherⁿ, whether they be prisms, cylinders, or cylinderoids, that is, whether the sections themselves be right-lined figures, circles, or curvilinear figures of any other kind.



Now if these sections HH, II, &c. be supposed to decrease, from the base upwards, so that the solid (HNNH, IOOI, &c.) formed upon them, may exceed the correspondent parts (HIIH, IKKI, &c.) of the given solid HAH; it is manifest, that the sum of all the said solids (HNNH+IOOI, &c.) will likewise exceed the whole proposed solid HAH^o. But, within HAH, on the same sections (but on contrary sides) another series of such solids IRRR, KSSK, &c. be formed; the sum of all these will manifestly, be less than the proposed solid HAH, in which they are contained^p. And it is also evident that this last series will be less than the former (HNNH+IOOI, &c.) by the greatest of these solids HNNH; because (this one, alone, being excepted to every other solid of the rank, an equal, in the contained rank, may be assigned, and *vice versa*: For 20. 7. and Sch. 5. 8. $IRRI=IOOI$, $KSSK=KPPK$, $LTTL=LQQL$.

Now, since the altitude BC, of the solid HNNH by which the contained, and containing series, differ from each other, may be taken so small a part of BC that the solid itself shall be less than any assigned magnitude whatever^r; it is manifest (from the reason

^r Lem. 1. and 22. 7.

ing in Corol. to Theor. I.) that a magnitude, which is greater than any series of solids (of the kind above specified) that can be formed within the proposed solid HAH, and less than any series that can be formed about HAH, must be equal to HAH. But the solid *hah*, being greater than any series of solids (*irri* + *kssk*, &c.) contained therein^s, is therefore greater Ax. 2. than any series of solids *IRRI* + *KSSK* &c. contained in HAH (*these*, being, respectively, equal to *those* :) And the same solid *hah*, being less than any series of solids (*hnnh* + *iooi*, &c.) formed about it, is also less than any series of solids (*HNNH* + *IOOI*, &c.) that can be formed about HAH. Therefore the solid *hah* is equal to HAH.

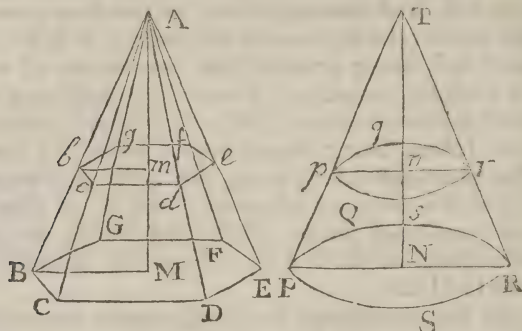
In this demonstration, the sections are supposed to decrease, continually, from the bases upwards; so as to have the sides of the upright solids formed thereon, placed wholly without, or wholly within, the superficies of the given solids HAH, *hah*; Which can only be the case, when all perpendiculars, from any points in the surface of either, to the plane of the base, fall within the limits of the base. If, however, the sections be supposed to decrease to a certain distance, only, and then to increase again; the two solids will, still, appear to be equal; Because the parts of the one, terminated by such limits of decrease, or increase, will (by the same demonstration) be respectively equal to the correspondent part of the other. But, as no such solids have a place in the *Elements of Geometry*, to say more about them here, would be improper.

LEMMA 5.

If pyramids and cones (ABCDEFGH, TPQRS) having equal altitudes (AM, TN,) be cut by planes parallel to the bases; the sections (bcdefg, pqrs) at all equal altitudes (Mm, Nn,) will be in the same proportion as the bases.

For the plane *bcdefg* being parallel to *BCDEFG*, hence is *bc* parallel to *BC*^a, *bg* to *BG*, &c. and consequently the angle *cbg* = *CBG*^b, *bcd* = *BCD*, &c. also^c 11. 7. 9. 7.

^e 14. 4. $bc : BC (:: Ab : AB^e) :: bg : BG$. And, in the same manner, the sides about the other equal angles are proportional. Therefore, the two polygons $bcdefg$



^d Def. 14. $BCDEFG$ being similar ^d, they are in proportion ^e, as bc^2 to BC^2 , or as Ab^2 to AB^2 , or, lastly, as Am^2 to AM^2 ; because (BM and bm being drawn) the angles AMB , amb will be right-ones ^g, and bm , therefore, parallel to BM ^h.

^h 4. 1. But the section $pqrs$ is also to the base $PQRS$ in the same proportion of Am^2 (Tn^2) to AM^2 (TN^2 ;) because $pqrs$ $PQRS$, being ⁱ circles, they are as the squares of their radii pn , PN ^k, and consequently as Tn^2 to TN^2 . Therefore, seeing that the two sections have both the same ratio to their respective bases, the proposition is manifest ^m.

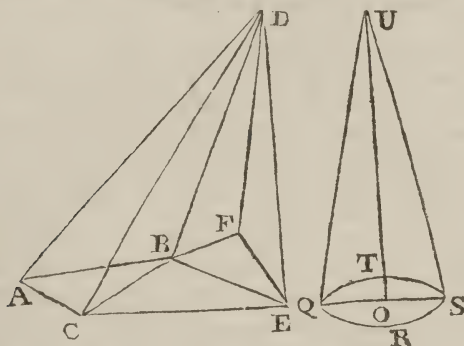
COROLLARY.

It appears from hence, that the section of any pyramid by a plane parallel to the base, is similar to the base.

THEOREM VI.

All pyramids of the same altitude standing upon equal triangular bases, are equal among themselves; and every such pyramid ($\triangle ABCD$) is equal to a cone ($QRSTU$) of equal base and altitude.

CASE I. If the perpendicular, let fall from the vertex D of the pyramid upon the plane of the base ABC, falls not out of the base, or beyond the limits of the triangle: Then it is manifest, from Lemma 4, seeing the sections of the solids ABCD, QRSTU, at all equal



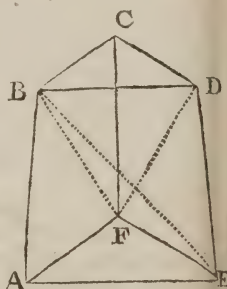
distances from the bases, will be equal^t, that the solids^t Lem. 5. themselves will likewise be equal.

CASE II. If the perpendicular (DE) from the vertex to the plane of the base, falls beyond the limits of the triangle: Then, to the point E where it meets the plane, let BE and CE be drawn; and on BE let a triangle EBF be described equal to ABC (or QRST,) and let F, D be joined. So shall the pyramid CBFED, standing on the base CBFE, be equal to the pyramid CABED, standing on the equal base CABE^u; from^u Lem. 4. each of which, let the common pyramid CBED be taken away; and there will then remain the pyramid BCFED=pyramid ABCD^w: But the former of these^w Ax. 5. is (by case 1.) equal to the cone QRSTU; therefore it is evident, that the latter ABCD will also be equal to the^x cone QRSTU; and, consequently, that all^x Ax. 1. pyramids of the same altitude, standing on equal triangular bases, will be equal among themselves^x; seeing every such pyramid is equal to a cone (QRSTU) of equal base and altitude.

THEOREM VII.

Every prism (ABCDEF) having a triangular base (AFE) is equal to the triple of a pyramid of the same base and altitude.

In the planes of the three sides, let the diagonals BE, BF, FD, be drawn. Then will the part FBCD of the prism cut off by a plane extended by FB and FD, be a pyramid on the base BCD, having the same altitude with the prism itself^y, both solids being contained between the same^z parallel planes AFE, BCD: Moreover, the remaining part FABDE of the prism, if a plane be extended by FB and FE, will be divided into the two pyramids FBAE, FDBE, which are equal to each other^a, as standing on the equal^b triangular bases ABE, BDE. But the former of these pyramids FBAE, if B be now considered as its vertex, will appear, also, to be equal to the first-mentioned pyramid FBCD^a, the two bases AFE, BCD (as well as the altitudes) being equal^b. Therefore, since the three triangular pyramids (FBCD, FBAE, FDBE) into which the prism is resolved, are all equal to each other; the proposition is manifest.



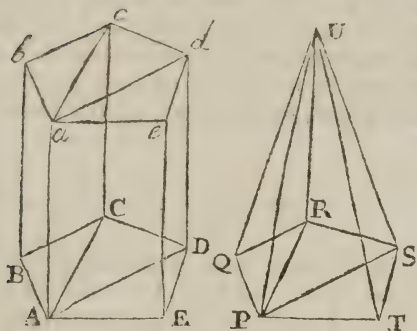
COROLLARY.

Hence, every prism having a triangular base, is equal to the triple of any pyramid of the same altitude, standing upon an equal triangular base^c.

^c 6. 8. and
Ax. 1.

THEOREM VIII.

If a prism (ABCE) and a pyramid (PQRSTU) stand upon equal and similar bases (ABCDE, PQRST) and have both the same altitude; the prism will be equal to the triple of the pyramid.



If the bases be resolved into triangles ABC , ACD , &c. it is manifest, that $BbacCA$ will be a prism, on the base ABC ; because Cc being equal and parallel to Aa $AacC$ will be a parallelogram^e (as well as ^d $Def. 6.$ $BbaA$ and $BbcC^f$). Therefore $BbacCA$ is equal to ^{and 8. 7.} the triple of the pyramid $PQRU^g$, standing on an ^{e 26. 1.} equal ^{Def. 6. 7.} base (PQR). And, in the same manner, the ^{s Cor. to} prism $AacdDC$, on the base ACD , is equal to the ^{7. 8.} triple of the pyramid $PRSU$, on the equal base PRS ; ^{h Hyp. and} and so on. Therefore, also, shall the whole prism Ad , ^{Ax. 10.} on the base $ABCDE$, be equal to the triple of the whole pyramid $PQRSTU$, on the equal base $PQRST$.

COROLLARY I.

Hence, all pyramids having the same base and altitude, are equal; being like parts of one and the same prism.

COROLLARY II.

Hence, also, all prisms having the same base, and altitude are equal; being equimultiples of one and the same pyramid.

COROLLARY III.

Therefore it appears, that every prism inclining on its base, as well as every upright one, is equal to a rectangular parallelepipedon of equal base

* 20. 7.
and Ax. 1.

and altitude^k; and, consequently, that *all prisms whatever, having equal bases, and altitudes, are equal to each other^k*: which must be also true in pyramids and cones, every such solid being subtriple to a prism, or cylinder, of the same base and altitude^l.

† 8. 8. and
6. 8.

COROLLARY IV.

Hence it also follows, that *whatever is demonstrated (in the 21st, 22nd, and 23rd Theorems of the preceding Book,) concerning the proportion of prisms, holds equally in pyramids and cones; these being like parts of those^l*.

COROLLARY V.

It follows, moreover, that *all corresponding frustums of pyramids and cones of the same altitude, are also, in proportion, as their bases*. For, the sections at all equal altitudes, being in that proportion^m, the parts cut off (as well as the wholes) will be in the same proportionⁿ; and consequently, the remaining parts likewise^o.

^m Lem. 5.

ⁿ Cor. 4.

^o 3. 4.

COROLLARY VI.

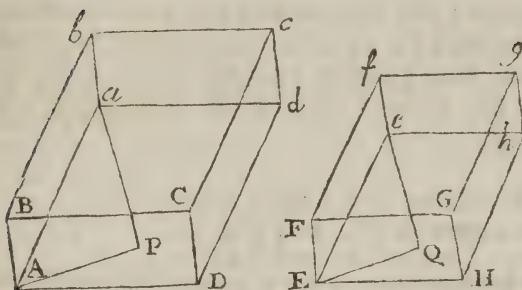
Lastly, it will appear, that *all cones, which have their altitudes and the diameters of their bases directly proportional, are in the triplicate ratio of their altitudes^p*; being to each other in the same proportion with prisms of equal base and altitude, of which they are like parts^q.

^p 24. 7.

^q Cor. 3.
and Cor.
to 1. 4.

THEOREM IX.

All similar prisms, and pyramids, are in the triplicate ratio of their altitudes.



From the extremes of the homologous sides Aa , Ee , upon the bases $ABCD$, $EFGH$ of the proposed solids Ac , Eg , let fall the perpendiculars aP , eQ . The angle BAD being $= FEH$, $BAa = FEE$, and $DAa = HEE$, thence is $aP : eQ :: aA : eE :: AB : EF :: AD : EH$.^{r Def. 5. 7.} Therefore two upright prisms constituted on the bases $ABCD$, $EFGH$, of the same altitudes (aP , eQ) with^{q 14. 7.} the two solids Ac , Eg , will be similar, the one to the other^{r Def. 14. 4.}, and, therefore, in the triplicate ratio of the altitudes^s. But the solids Ac , Eg , when taken as^{r Def. 5. 7.} prisms, are respectively equal to the said upright ones; and, when taken as pyramids, are like parts of them^{t. Cor. 3. to 8. 8.}. Therefore the solids Ac , Eg , are also in the triplicate ratio of the altitudes aP and eQ .^{u Cor. 1. to 5. 4.}

COROLLARY.

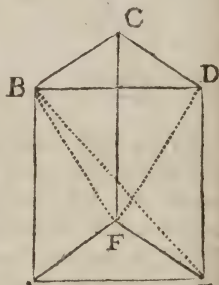
Because $aP : eQ :: Aa : Ee :: AB : EF$, &c. it follows, that *all similar prisms, and pyramids, are to one another, in the triplicate ratio of the homologous sides of the like planes by which they are bounded*.^{v Cor. 1. to 5. 4.}

THEOREM X.

The frustum $(ABCDEF A)$ of any pyramid having a triangular base, is equal to a whole pyramid, of the same base and altitude, together with two other pyramids, that are in proportion

thereto; the one, as any side (BD) of the upper base (BCD) is to its correspondent (AE) of the lower base (AFE;) and the other, as the square of the former side is to the square of the latter.

In the planes of the three sides let the diagonals BE, BF, FD be drawn. Then will the part FBCD of the frustum, cut off by a plane extended by FB and FD, be a pyramid, on the base BCD, having the same altitude with the frustum itself^z, both solids being contained between the same^a parallel planes AFE, BCD.



^z Def. 16.

^{7.}

^a Def. 15.

^{7.}

Moreover, the remaining part FABDE of the frustum, if a plane be extended by FB and FE, will be divided into the two pyramids FBDE, FAFE, having the same ratio, one to the other, as their bases BED, ABE^b, or as BD to AE^c. But the latter of these pyramids (BAFE,) taking B as its vertex, has the same base and altitude with the frustum given: And the pyramid FBCD (first mentioned) is therefore, in proportion thereto, as the base BCD to the base AFE^d, that is, because the bases are similar^e), as BD^2 to AE^2 : whence the proposition is manifest.

^b 21. 7. and

^{Cor. 4. to}

^{8. 8.}

^c 7. 4.

^d 21. 7. and

^{Cor. 4. to}

^{8. 8.}

^e Cor. to

^{Lem. 5.}

^f 26. 4.

COROLLARY.

Since, of the three solids (FAFE, FBDE, FBCD) into which the proposed frustum is divided, the ratio of the first and third, is the duplicate of that of AE to BD, or of the ratio of the first to the second^g; it is evident, that these three solids are proportionals^h. From whence it appears, that the frustum of any triangular pyramid is equal to two (whole) pyramids of the same altitude, on bases equal to the two opposite bases of the frustum, and to a third pyramid, which is a mean proportional between the two former. And it is also evident,

^g Cor. to

^{27. 4.}

^h Def. 7. 4.

that whatever is above demonstrated, in relation to triangular pyramids, must hold equally in all pyramids and cones, whatever; Because every such solid is equal to a triangular pyramid, of equal base and altitudeⁱ; and every frustum of the one, also equal to the corresponding frustum of the other^k.

ⁱ Cor. 3. to 8. 8.
^k Cor. 5. to 8. 8.

LEMMA 6.

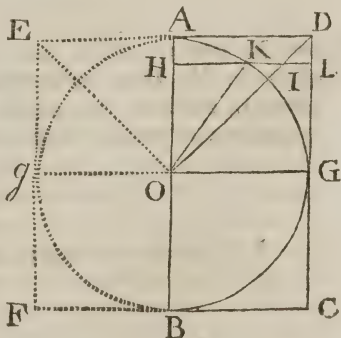
If with radii, respectively equal to the three sides of any right-angled triangle, three circles be described; that whose radius is equal to the hypotenuse, will be equal to both the other two, taken together.

It has been proved (*in Theor. 3.*) that circles are in proportion, as the squares of their radii; therefore the demonstration here, is the very same, as in similar right-lined figures (*Theor. 29. Book 4.*): Which (if necessary) you may re-consult.

THEOREM XI.

Every sphere is two-thirds of its circumscribing cylinder (Or, of a cylinder of equal diameter and altitude.)

Let AB be the axis about which the sphere and cylinder are generated, by the revolution of the semi-circle AGB and the rectangle ADCB^l; let HL be any right-line perpendicular to AB, meeting DC in L, and the periphery of the semicircle in K; and from the center O, let OK and OD, intersecting HL in I, be drawn.



^l Def. 12. and 14.

^m Hyp. Since AO is $= AD^m$, and HI parallel to AD^n , there-
ⁿ 4. 1. fore is $HI = OH^o$: But OHK being right-angled at
^o 14. 4. H , the circle whose radius is OH (or HI) will (*by the preceding Lem. and Ax. 5.*) be equal to the difference of the two circles whose radii are OK (HL) and HK : Or, in other words, the circle^p described by HI , or the section of the cone generated by the triangle AOD , in its revolution about the axis AB , will be equal to the difference of the two circles generated by HL and HK ; that is, equal to the
^r Lem. 6. *annulus* described by KL^q , or the section of the solid which remains, when the sphere is taken out of the cylinder. Therefore, seeing these two sections are, every where, equal to each other, the solids them-
^r Ax. 2. 1. selves will likewise be equal^r; that is, the cone (EOD) will be equal to the excess of the cylinder ($GDEg$) above the inscribed hemisphere (GAg): whence, as
^r Lem. 4. the cone, or excess, is one-third part of the cylinder^s the hemisphere must necessarily be equal to the two remaining thirds. And what is here proved, with respect to the halves of the proposed solids, holds equally in the wholes. Therefore every sphere is two-thirds of its circumscribing cylinder.
^s 6. 8. and
^{Cor. 3. to}
^{8. 8.}

COROLLARY I.

Hence, *a cone, hemisphere and cylinder, of the same altitude, and standing upon equal bases, are in proportion, as the numbers 1, 2, and 3, respectively.*

COROLLARY II.

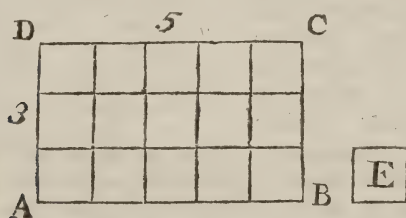
^r Cor. to 5. Hence it also appears, that *all spheres are to each other in the triplicate ratio of their diameters^t*;
^{8. and 24.} being in the same proportion as the circumscribing
^{7.} cylinders, of which they are like parts.

OF THE MENSURATION

OF

SUPERFICIES AND SOLIDS.

MENSURATION OF SUPERFICIES.



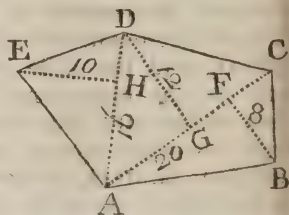
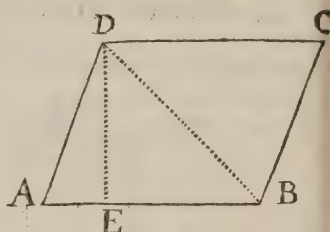
EVERY quantity is measured by some other quantity of the same kind ; as a line by a line, a surface by a surface, and a solid by a solid : And the number which shows how often the less, called the *measuring unit*, is contained in the greater, or quantity measured, is called *the content of the quantity so measured*. Thus, if the quantity to be measured be the rectangle ABCD, and the little square E, whose side is one inch, be the measuring unit propounded ; then, as often as the said little square is contained in the rectangle, so many square inches the rectangle is said to contain : So that, if the length DC be supposed 5 inches, and the breadth AD 3 inches ; the *area*, or su-

perficual content of the rectangle will be 3 times 5, or 15 square inches: Because, if lines be drawn parallel to the sides, at an inch distance one from another, they will divide the whole rectangle ABCD into 3 times 5, or 15 equal parts, of one inch each.

And, *generally*, whatever the measures of the two sides may be, it is evident (*from El. 7. of 4.*) that the rectangle will contain the square E, as many times as the base AB contains the base of the square, repeated as often as the altitude AD contains the altitude of the square. Therefore, *to find the superficial content of any rectangle, multiply the base by the altitude, and the product will be the answer.* Thus, let the length be 18 inches, and the breadth 15; then the content will be 15 times 18, or 270 square inches.

The method of finding the content of a rectangle being thus known,, the content of any parallelogram ABCD, or triangle ABD, will also be known; the former of these figures being equal to a rectangle of the same base and altitude; and the latter equal to the half of such a rectangle (*by Cor. 2. to 2, 2.*). Therefore, *multiply the base by the perpendicular, for the content of any parallelogram; and the base by half the perpendicular, for that of any triangle.* Thus, for example, let the base AB be 18 feet, and the perpendicular DE 12 feet; then the content of the parallelogram will be 216, and that of the triangle 108 square feet.

From the manner of finding the area of a triangle, the area of any right-lined plain figure, as ABCDE, may be determined, by dividing the whole into triangles, and finding the content



of each triangle. Thus, let the dividing lines AC and AD, be 20 and 16 inches, and the perpendiculars BF, DG, EH, falling thereon, 8, 12, and 10, respectively; then the content of the

$$\text{Triangle} \left\{ \begin{array}{l} ABC \\ ACD \\ ADE \end{array} \right\} \text{ being } \left\{ \begin{array}{l} 80 \\ 120 \\ 80 \end{array} \right\}, \text{ it is evi-}$$

ent, that the content of the whole figure will be the sum of all these, or 280 square inches. But, when the given lines are expressed by fractions, or very large numbers, the work will be somewhat shortened, by finding the content of every two triangles, having the same base (forming what is called a *Trapezium*,) at one operation; that is, first *add the two perpendiculars together, and then multiply half their sum by the common base of the two triangles*. Thus, in the last example, the half-sum of the two perpendiculars BF and DG being 10, if this number be, therefore, multiplied by 20, the measure of the common base AC, the product which is 200, will be the content of the trapezium AECDA; to which 80, the content of the triangle ADE being added; the sum will be 280, the same as before.

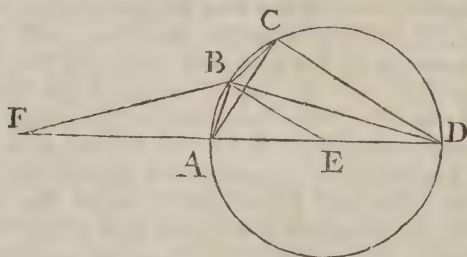
But, if the polygon proposed be a regular one, that is, one whose sides, and angles are all equal, the shortest way of all, is, to *multiply half the sum of all the sides by the length of the line drawn from the middle of any side to the center of the polygon*. Thus, suppose it were required to find the area of a pentagon, whose side is 250, and perpendicular distance of the center from the middle of any side, 172 inches;—Here $250 \times 5 = 1250 =$ the perimeter, or sum of the sides; then $\frac{1250}{2} \times 172 = 625 \times 172 = 107500$ the superficial content of the pentagon, in square inches. The reason of which is obvious, from the demonstration to *Theor.* P. B. VIII.

Having shown how the area of any right-lined figure may be computed, it will be proper here, to say something with regard to the area, and periphery of the circle.

It is well known, that to determine the *true* area of a circle, and to find a right-line *exactly* equal to its circumference are looked upon, by mathematicians as absolutely impossible: But, though neither the one nor the other can be *accurately* known, yet several ways have been invented by which they may be approximated, to any assigned degree of exactness. That which I am now going to lay down, though less expeditious than some others, seems, nevertheless, to be the most proper for this place, as depending on the most simple and evident principles: I shall therefore begin with premising the following

LEMMA.

If AD be a diameter, and AB, BC two equal arcs of the same circle, and if the chords DB, DC be drawn; then $DB^2 = \frac{1}{2}AD \times DC + \frac{1}{2}AD^2$.



For, if in DA produced, there be taken $AF = DC$ and BF, BA, BC and the radius BE be drawn; then the external angle FAB , of the trapezium $ABCD$ being equal to the internal opposite angles DCB (*by* 17, 3.) also $AF = DC$, and $AB = CB$ (*by Hyp.*); it is evident, that FB is also $= DB$, and consequently the angle $F = FDB = DBE$: And so the isosceles triangles DEB, DBF being equiangular, it will be as $DE (\frac{1}{2}AD) : DB :: DB : DF (DC + AD)$; and consequently $DB^2 = \frac{1}{2}AD \times DC + \frac{1}{2}AD^2$. *Q. E. D.*

COROLLARY.

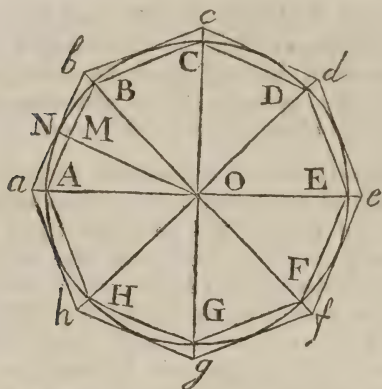
Hence, if the diameter AD be denoted by the number

2, the chord DB will be denoted by $\sqrt{DC+2}$: whence, it appears, that, *if the measure of the supplemental-chord of any arch be increased by the number 2, the square-root of the sum will be the supplemental-chord of half that arch.*

Now, to apply this to the matter proposed, that is, to the finding of the area and circumference of the circle; let the arch ABC be taken equal to $\frac{1}{3}$ of the semi-periphery ACD; then will the chord AC be equal to the radius AE (*by 29. 5.*); and therefore, since ACD is a right angle (*by 13. 3.*) $DC^2 (=AD^2 - AC^2, \text{ by } 8. 2.)$ will be $=4-1=3$; and consequently $DC=\sqrt{3}=1,7320508075, \&c.$ Wherefore, seeing the supplemental chord of $\frac{1}{3}$ of the semi-periphery is 1,7320508075, we shall by the preceding corollary,

$$\left. \begin{array}{l} \sqrt{2+1,7320508075}=1,9318516525 \\ \sqrt{2+1,9318516525}=1,9828897227 \\ \sqrt{2+1,9828897227}=1,9957178465 \\ \sqrt{2+1,9957178465}=1,9989291743 \\ \sqrt{2+1,9989291743}=1,9997322757 \\ \sqrt{2+1,9997322757}=1,9999330678 \\ \sqrt{2+1,9999330678}=\sqrt{3,9999330678} \end{array} \right\} \begin{array}{l} \text{have} \\ \\ \\ \\ \\ \\ \end{array} \left. \begin{array}{l} \sqrt{2+1,7320508075}=1,9318516525 \\ \sqrt{2+1,9318516525}=1,9828897227 \\ \sqrt{2+1,9828897227}=1,9957178465 \\ \sqrt{2+1,9957178465}=1,9989291743 \\ \sqrt{2+1,9989291743}=1,9997322757 \\ \sqrt{2+1,9997322757}=1,9999330678 \\ \sqrt{2+1,9999330678}=\sqrt{3,9999330678} \end{array} \right\} \begin{array}{l} \text{for the supplemental} \\ \text{chord of} \\ \\ \\ \\ \\ \end{array} \left\{ \begin{array}{l} \frac{1}{6} \\ \frac{1}{12} \\ \frac{1}{24} \\ \frac{1}{48} \\ \frac{1}{96} \\ \frac{1}{192} \\ \frac{1}{384} \end{array} \right\} \begin{array}{l} \text{of the periphery.} \\ \\ \\ \\ \\ \\ \end{array}$$

Now, therefore, since it is found that 3,9999330678 is the square of the supplemental-chord of $\frac{1}{384}$ of the semi-periphery, let this number be subtracted from 4 the square of the diameter, and the remainder 0,0000669322 will be the square of the chord of the same arch; therefore the chord itself being $=\sqrt{0,0000669322}=0,00818121$, let this number be multiplied by 768, or twice 384, and the product 6,28317 will be the perimeter of a regular polygon of 768 sides, inscribed in the circle; which, as the sides of the polygon very nearly coincide with the circumference of the circle, must also express the length of the circumference itself, very nearly.



But, in order to show how near this is to the truth, let AB represent one side of a regular polygon of 768 sides, inscribed in the circle (whose length, we have found above to be 0,00818121) and let ab be a side of another similar polygon, described about the circle; and from the center O , let ON be drawn, bisecting AB and ab in M and N : Then, since AM is $=\frac{1}{2}AB = 0,0040906$, and $AO=1$, it is plain, that $OM^2 (AO^2 - AM^2)$ will be $=0,99998327$, and consequently $OM = 0,99999163$; whence, because of the similar triangles AOB , aOb , &c. we have $0,99999163 (OM) : 1 (ON) :: AB : ab :: 6,28317$ (the perimeter of the inscribed polygon) : $6,28322$ the perimeter of the circumscribed polygon. But the circumference of the circle being greater than the perimeter of the inscribed polygon, and less than that of the circumscribed one, it must, consequently, be greater than $6,28317$, and less than $6,28322$; and must, therefore, be equal to $6,2832$, very near, since this number exceeds the perimeter of the inscribed polygon by no more than $0,00003$, and is less than the perimeter of the circumscribed one by $0,00002$, only.

From the periphery thus found, the area of the circle will also be known; being equal to the product of half the periphery into the radius (*by 2. 8.*) that is, $=3,1416 \times 1 = 3,1416$.

Therefore, since it is proved (*in Theor. 3, and 4, of 8.*) that the peripheries of circles are in proportion as their

diameters, and the circles themselves as the squares of those diameters; it follows, that, as 2 is to 6,2832, or as 1 to 3,1416 :: the diameter of any circle to its periphery; and as 4 to 3,1416, or as 1 to 0,7854 :: the square of the diameter to the area.

But, if you would rather have the proportions in whole numbers, and the case proposed does not require any great degree of accuracy, then, instead of the foregoing, those of *Archimedes* may be used, *viz.* 7 : 22 :: diam. : circumf. and 14 : 11 :: square diam. : area. Which proportions differ but little from those above, as will appear from the following example: wherein the diameter of a circle being given 28, its circumference and area are required. Here, according to the first proportions, I multiply 28 by 3,1416, for the circumference, and the square of 28 (or 784) by 0,7854 for the area; and there results 87,964 and 615,75, respectively. But, according to the proportions of *Archimedes*, the circumference will be found equal to 88, and the area 616; which differ very little from the former.

By knowing the proportion between the diameter of a circle and the circumference, and between the square of the diameter and the area, the convex superficies of solid bodies may be determined. Thus,

The convex superficies of a cylinder is found, *by first finding the circumference of the base, and then multiplying by the altitude of the solid.* Therefore, if to that product, the area of the two circular ends be added, the sum will be the whole superficies of the cylinder.

To find the convex superficies of a cone, *multiply half the length of the slant side by the circumference of the base.*

The convex superficies of any frustum of a cone is found, *by multiplying the sum of the peripheries of the two ends into half the length of the slant side of the frustum.*

To find the superficies of a sphere, *multiply the periphery of the greatest, or generating, circle by its*

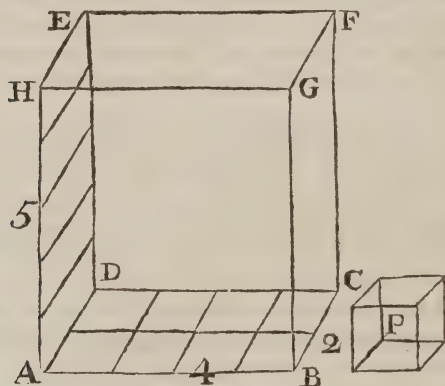
diameter: Or, multiply the square of the diameter by 3,1416.

The convex superficies of any segment of a sphere is found, by *multiplying the periphery of the greatest circle of the sphere into the altitude of the segment.*

The demonstration of these last rules; for finding the curve surfaces of solid bodies (which is not given in the *Elements*, for reasons mentioned hereafter) is inserted at the end of this section.

OF THE MENSURATION OF SOLIDS.

AS every superficies is measured by a square, whose side is unity (as one inch, one foot, one yard, &c.) so every solid is measured by a cube, of which the side is also an unit. Thus, let the solid to be measured,



be the rectangular parallelepipedon AF, and let the cube P, whose side is one inch, be the *measuring unit*; also let the length AB, of the base AC, be 4 inches, the breadth BC 2 inches, and the altitude AH of the solid 5 inches: Then, because the area of the base ABCD is 2 times 4 (or 8) square inches, it is

easy to conceive, that, if the solid were to be only one inch high (instead of 5,) the content would be just the same number (8) of cubical inches; because then, upon each of the eight equal squares into which the whole base ABCD is divisible, a cube of one inch might be erected, so as to compose a parallelepipedon on that base, of one inch high. Therefore, seeing that the content of the solid, at one inch high, is 8 cubical inches, the whole content at 5 inches high, must consequently be 5 times 8, or 40 cubical inches (since the whole solid AF may be considered, as composed of 5 such heights of cubes, one ranged above another.)

And, generally, whatever the dimensions may be, it is manifest (*from 21 and 22. of 7.*) that the parallelepipedon will contain the cube P, as many times as the base ABCD contains the base of the cube, repeated as often as the altitude AH contains the altitude of the cube.

Therefore *the content of any parallelepipedon will be found, by multiplying the area of the base by the altitude of the parallelepipedon.* Thus, for example, if the two dimensions of the base be 16 and 12 inches, and the height of the solid 10 inches; then, the area of the base being 192, the content of the solid will be 920 cubical inches.

From the content of a *parallelepipedon*, thus known, that of a *prism*, or a *cylinder*, will likewise be known; every such solid being (*by 20. 7. or 5. 8.*) equal to a parallelepipedon of equal base, and altitude. Therefore, *multiply the area of the base* (found by the rules for superficies) *into the height of the prism, or cylinder, and the product will be the content.*

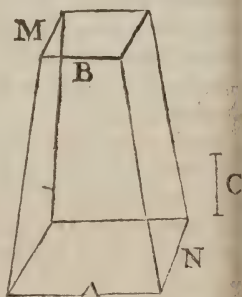
Hence the content of any *pyramid*, or *cone*, is also obtained; being (*by Cor. 3. to 8. 8.*) equal to $\frac{1}{3}$ part of a prism, or cylinder, of the same base and altitude. Therefore, *multiply the area of the base by $\frac{1}{3}$ of the altitude, and the product will be the answer.*

Every *sphere* being (*by 11. 8.*) equal to $\frac{2}{3}$ parts of a cylinder of the same diameter and altitude; *the content of any sphere will, therefore, be found, by multiplying the area of its greatest, or generating, circle*

into $\frac{2}{3}$ of its diameter : Or (because the area of such circle is to the square of the diameter, in proportion as 0,7854 to 1,) let the cube of the diameter be multiplied by the fraction ,5236 ($=\frac{2}{3}$ of 0,7854,) and the product will be the content. Thus, if the measure of the diameter be 20, its cube will be 8000 ; which, multiplied by ,5236, will give 4188,8 for the measure of the sphere's solidity.

The manner of finding the content of any *frustums* of the solids above determined, is collected from *Theor. 10. and 11. B. VIII.*

Let the frustum (MN), first proposed, be that of a *pyramid*; then, having found the content of a whole pyramid, of the same given base and altitude; say, as any side A of the lower end or base, is to its correspondent B of the upper, so is the said content of the whole pyramid to a fourth proportional.

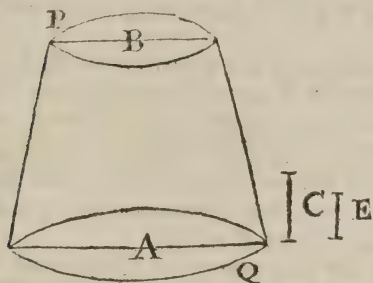


And, as A is (again) to B, so is the quantity last found to another proportional: which two proportionals, added to the content first determined, will give the true content of the frustum.

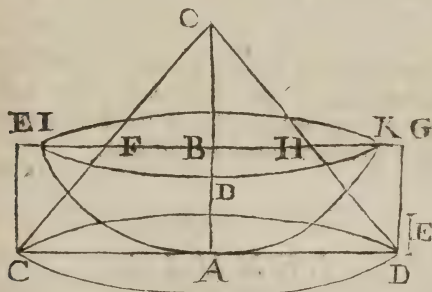
But when the opposite bases of the frustum are squares, the rule will be more simple, and put on a better form : For then the area of the base being A^2 , the content of a whole pyramid thereon, of the same altitude with the frustum, will be equal to the parallelepipedon $C \times A^2$, C being $\frac{1}{3}$ of the given altitude of the frustum.

But $A : B :: C \times A^2 : C \times A \times B$ (by 22. 7.) and $A : B :: C \times A \times B : C \times B^2$. Therefore $C \times A^2 + C \times A \times B + C \times B^2 (=C \times \overline{A^2 + A \times B + B^2})$, by *Schol. to 20. 7.* is the true content, in this case.

Hence, to find the content of the frustum of any square pyramid, add the product of the two sides of the lower and upper ends to the sum of their squares, and then multiply the aggregate by $\frac{1}{3}$ of the frustum's height.



From the content here found, *that* of any conical frustum (PQ) is readily obtained; being in proportion to the content $(C \times A^2 + A \times B + B^2)$ of the frustum of a square pyramid circumscribing it, as the base of the former is to the base of the latter (*by Cor. to 8. 8.*), or as the fraction ,7854 is to unity: And C , will be equal to the ,7854 part of $C \times A^2 + A \times B + B^2 = E \times A^2 + A \times B + B^2$; by taking $E = ,7854 \times C$ the ,2618 part of the whole given altitude. Therefore, to find the content of any frustum of a cone, add the product of the diameters of the two ends to the sum of their squares; then multiply the aggregate by the frustum's height, and the product, again, by the fraction ,2618.



Hence, and from *Theor. 11. B. VIII.* a rule for finding the content of any segment IAK of a sphere, may also be deduced. For, it appears, from thence, that the segment proposed, IAK, is equal to the difference between a conical frustum FCDH and a cylinder ECDG of the same altitude, standing upon a base,

whose radius CA is equal to that (AO) of the sphere itself.

But the content of the frustum FCDH, if the two diameters CD, FH be represented (*as above*) by A and B, and the ,2618 part of the altitude (D) by E will be = $E \times A^2 + A \times B + B^2$ (that is, equal to a parallelepipedon whose altitude is E, and base = $A^2 + A \times B + B^2$): And the content of the cylinder ECDG will be = $3E \times A^2$, or $E \times 3A^2$.

Therefore the difference (or the content of the segment IAK) will be = $E \times 2A^2 - A \times B - B^2$ (*Schol. to 20. 7. and Ax. 5. 1.*)

But $2A^2 - A \times B - B^2$ is composed of $A^2 - A \times B$ and $A^2 - B^2$; the former part of which $A^2 - A \times B$ is = $A - B \times A$ (*by 5. 2.*) = $2D \times A$ (because $A - B$ (or $CD - FH$) = $FE + HG = 2AB$ or $2D$); and the latter $A^2 - B^2$ = $A - B \times A + B$ (*by 7. 2.*) = $2D \times 2A - 2D$.

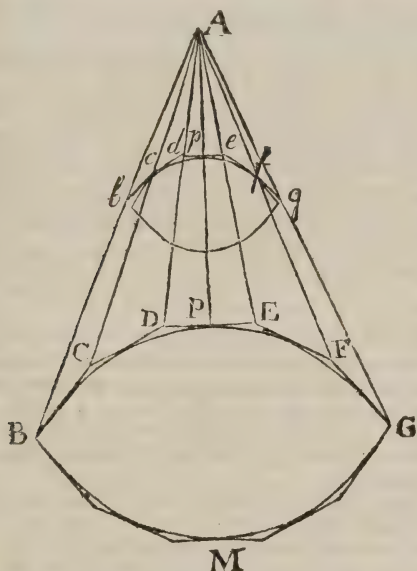
Whence the sum of both will consequently be = $2D \times 3A - 2D$; and the content of the segment itself = $E \times 2D \times 3A - 2D$ = ,5236 $\times D^2 \times 3A - 2D$ (because $2E$ = ,5236D).

Therefore, to find the content of any segment of a sphere, multiply the square of the segment's height by the excess of thrice the sphere's diameter above the double of that height; and then multiply by the fraction ,5236.

The demonstration of the rules for determining the superficial content of the cylinder, cone and sphere and of their several segments, or frustums, is collected from the two Lemmas here subjoined.

LEMMA 1.

The upper superficies, or area of all the sides of a regular pyramid, in which a cone may be inscribed, is equal to a rectangle under the perimeter of the base, and half the length of the cone's slant side.



For, let BCDE, &c. be the base of the pyramid, and BPGM that of the inscribed cone; and from the vertex A to the point P where any side DE of the polygon touches the circle, let AP be drawn. Then, since the triangle ADE is $= \frac{1}{2} AP \times DE = \frac{1}{2} AB \times DE$; and as the like holds good with regard to every other side of the pyramid, it is evident, that the sum of all the sides, or the whole superficies of the pyramid (exclusive of the base) will be equal to $\frac{1}{2} AB \times DE + EF + \&c.$; that is, equal to a rectangle under $\frac{1}{2} AB$ and the whole perimeter of the base.

COROLLARY I.

Hence it will also appear, that all the sides of any frustum Bg of the pyramid, will be equal to a rectangle under half the length of each side and the sum of the perimeters of the two ends: For, the area of the side DEed being $= \frac{1}{2} Pp \times DE + de$, or $\frac{1}{2} Bb \times DE + de$ (by 4. 2.), the area of all the sides will, therefore, be $= \frac{1}{2} Bb \times DE + de + EF + ef + \&c.$

COROLLARY II.

Therefore, seeing that the foregoing conclusions hold universally, whatever the number of the sides may be; and as the pyramid, by increasing the number of its sides, approaches nearer and nearer, continually, to the inscribed cone, which is its limit; thence will the upper or *convex superficies* of the cone (as well as that of the pyramid) be equal to a rectangle under half the length of its slant side, and the perimeter of its base.*

* The rule above given, applies only to a *right* cone, that is, to a cone, whose axis is perpendicular to its base; no method has yet been discovered of determining, *accurately*, the curve surface of an *oblique* cone, though many attempts have been made, at different times, by some of the most eminent mathematicians in Europe: the following are among the principal which have come under the Editor's observation:

1st. In 1648, *Richard White*, an Englishman, published a work at Rome, in 4to, entitled, "*Hemisphaerium dissectum, opus geometricum accessit appendix, de inscriptione in sphaera conicali, et de superficie ejus*" in which he shows that the curve surface of an oblique cone is nearly equal to the area of a circle, the radius of which is a mean proportional between $\frac{1}{2} \sqrt{(h^2 + r^2)} + \frac{1}{4}S + \frac{1}{4}s$; where r denotes the radius of the cone's base, h the perpendicular height, and S and s the greatest and least slant sides. See *Leybourn's Mathematical Repository*, N. S. Vol. 1. page 113.

2nd. In the *Miscell. Berol.* Tom. III. 1727, *Peter Varignon* published his "*Schediasma de superficie conĩ ad basin circulařen obliquĩ ope longituninis curvę*" with an "*additio . . .*" by *Leibnitz*.

3rd. In the *Nov. Comment, Acad. Petropol.* Tom. I. ad ann. 1747. 1748, is a memoir by the illustrious *Euler*, *de superficie conorum scalenorum aliorumque corporum conicorum* and in the *Nov. Act. Acad. Petropol.* Tom. III. 1785, is another memoir, by the same profound analyst, *de superficie conĩ scaleni, ubi imprimĩ ingentes difficultates, quę in hac investigatione occurrunt, perpenduntur*

4th. *D'Alembert*, in Tome I. of his *Opusculs Mathematiques* Paris, 1761, has a memoir, "*de la surface d'un cone, qui a pour base une ellipse . . .* pa. 236. avec un *addition au memoire precedent* pa. 244; and another, *de la surface des cones obliques*.

5th. In the *Comment. Soc. R. Gotting*, the celebrated *Abr. Gotting* has a memoir, *de conĩ scaleni superficie et reti*.

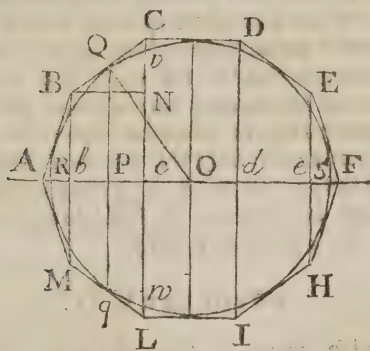
6th and lastly. In the *Ladies Diary* for 1753 (Q. 369), an approximation is given by Mr. *Thomas Simpson*, the author of the *Elements*; and the subject is again brought under consideration, in the same *Diary* for 1806 (Prize Q.)

And the convex superficies of any *frustum* of the cone will, also, be equal to a rectangle under half the length of its slant side, and the sum of the peripheries of its two ends, or bases.

Whence it likewise follows, that the convex surface of a *cylinder* will be equal to a rectangle under half its altitude, and twice the periphery of its base (or under the whole altitude, and once that periphery;) because then the two ends are equal.—From this Corollary, the rules for finding the superficies of the cylinder and cone, are given.

LEMMA 2.

If a regular polygon *ABCDE*, &c. of an even number of sides, together with its inscribed circle *RQsq*, be supposed to revolve about the (produced) diameter *RS*, as an axis; the superficies of the solid generated by the polygon, will be equal to a rectangle under its axis *AF* and a right-line equal to the circumference *RQsq* of the inscribed circle.



From the center *O*, to the point of contact *Q*, of any side *BC*, let the radius *OQ* be drawn; also draw *BbM*, *QPq*, *CcL*, &c. perpendicular to *AF*, and *BN* perpendicular to *CL*.

Because the solid generated by the plane $BbcC$ is the frustum of a cone, the convex superficies of that frustum generated by BC , is equal to a rectangle under $\frac{1}{2}BC$ and the sum of the peripheries of the two circles described by Bb and Cc , (*by Cor. 2. to the precedent*): But the sum of these two peripheries, as (QP is an arithmetical mean between Bb and Cc), is equal to twice the periphery Qq ; and therefore the convex superficies of the said frustum equal to $\frac{1}{2} BC \times 2$ periph. $Qq = BC \times$ periph. Qq . But, because of the similar triangles OPQ , BNC , we have $BC : BN (bc) :: OQ : PQ ::$ periph. $RQSq : \text{periph. } Qq$ (*by 4. 8.*); and consequently $BC \times$ periph. $Qq = bc \times$ periph. $RQSq =$ the superficies generated by BC . By the very same argument, the superficies generated by any other side CD is $= cd \times$ periph. $RQSq$: Whence it is manifest, that the superficies of the whole solid is $= \overline{Ab + bc + cd + \&c.} \times$ periph. $RQSq = AF \times$ periph. $RQSq$.

COROLLARY I.

Since the superficies of the solid is, universally, equal to $AF \times$ periph. $RQSq$, let the number of sides of the generating polygon be what it will; and as the said superficies, by increasing the number of sides, approaches nearer and nearer, continually, to the superficies of the inscribed sphere, which is its limit; therefore *the superficies of the sphere, itself, is equal to a rectangle under its axis RS and periphery $RQSq$; and the convex superficies of any segment of a sphere vRw , is likewise equal to a rectangle under its axis (or height) Rc and the same periphery $RQSq$* ; since it is proved, that the corresponding superficies of $CBAML$, is universally equal to $Ac \times$ periph. $RQSq$.

COROLLARY II.

Hence it also appears, that *the superficies of every sphere is equal to four times its generating circle*; Because (*by 2. 8.*) the circle $RQSq = \frac{1}{2} RS \times \frac{1}{2}$ periph. $RQSq = \frac{1}{4} RS \times$ periph. $RQSq$.

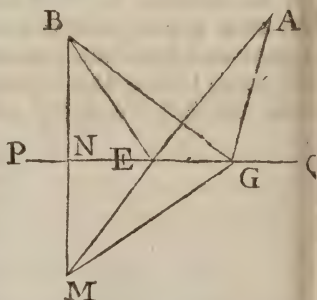
In deriving these conclusions, as well as those depending upon the preceding Lemma, the *Reader* must have observed, that something is assumed, which is not demonstrated in any part of these *Elements*. But this will not, I imagine, be considered as a fault, by those who know, that it is impossible to prove in a manner *perfectly* regular and geometrical, that a *curve surface*, of any kind, is equal to a *plane-one* of an assigned magnitude. Plane surfaces are compared with one another, in virtue of the 10th *Axiom*; in which, whatever relates to the equality of plane figures, has its original. But no principles have been yet admitted into the *common*, or *lower Geometry*, by which a curve-surface can be compared with a plane-one; nor even by which the proportion of any one curve-line to a right-line can be known: Nor can it be demonstrated by all the Geometry in *Euclid's Elements*, that the periphery of a circle is less than the perimeter of its circumscribing square.—We can determine the proportion of solids bounded by curve-surfaces, by describing other solids in, and about them, so as to differ less from them, than by any assigned part, however small. But, in comparing the surfaces, this method fails; because, let the number of sides of the inscribed, or circumscribed solid, be ever so great, or let the solid itself approach ever so near to the proposed one; the two surfaces, after all, *will have no part in common* on which a demonstration can be formed, but will still be distinct things. Before such a comparison can possibly be made, in a regular and scientific manner, *new principles* must be laid down: But *these* belong to, and are best supplied in the *Modern Geometry*, or *Method of Fluxions*.

OF THE
MAXIMA AND MINIMA
OF
GEOMETRICAL QUANTITIES.

THEOREM I.

If from two given points A, B, on the same side of an indefinite line PQ (in the same plane with them) two lines AE, BE be drawn to meet on, and make equal angles AEQ, BEP with the said line PQ; the lines so drawn, taken together, shall be less than any other two AG, BG drawn from the same points to meet on the same line PQ.

For, let BNM be perpendicular to PNQ, and let AE be produced to meet it in M, also let MG be drawn.



^a 3. 1.
^b Hyp.
^c Constr.

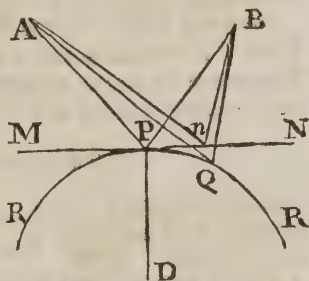
^d 15. 1.
^e Ax. 10. 1.
^f 19. 1.

Then the triangles MNE, BNE, having the angle MEN (=AEQ^a) = BEN^b, MNE=BNE^c and NE common; have also MN=BN, and ME=BE^d; whence also MG=BG^e: But AM (AE+BE) ^f AG + MG^f, or, than its equal AG + BG Q. E. D.

THEOREM II.

If all right-lines AP, BP; AQ, BQ, that can be drawn from two given points A, B, to meet, two by two, on the convexity of a given circle RPQR; those two AP, BP taken together, shall be the least, which make equal angles with the tangent MPN (or with the radius DB) at the point of concurrence P.

For, if to any point n in the part of the tangent intercepted by AQ and BQ, there be drawn An and Bn; then $AP + BP \supset An + Bn^g$, and $An + Bn \supset AQ + BQ^h$: Consequently $AP + BP \supset AQ + BQ$. Q. E. D.



^g Theor. 1.

^h 23. 1.

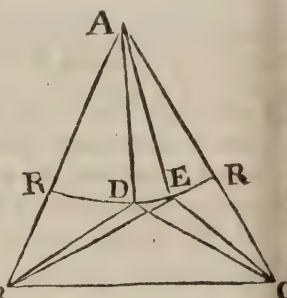
This demonstration holds equally true, when the curve RPR is supposed of any other kind; provided all tangents to it, fall entirely without the curve.

THEOREM III.

If, in a given triangle ABC, a point is to be determined, so that the sum of all the three lines drawn from thence to the three angles, shall be the least possible; I say, the position of that point must be such, that all the angles formed about it by those lines, shall be equal among themselves.

If you deny it, then let some point E, at which the angles BEA, CEA are unequal, be the required one.

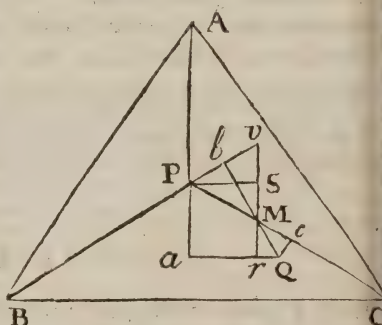
Upon the center A, through E, let the circumference of a circle RER be described; and let D be that point in it, where the angles ADB and ADC B are equal.



- ⁱ Theor. 2. Because $BD + CD \supset BE + CE$ ⁱ, therefore is $AD +$
^k Ax. 6. 1. $BD + CD \supset AE + BE + CE$ ^k; which is repugnant
 Therefore no point at which the angles are unequal
 can be the required one. Q. E. D.

The same otherwise.

Let the point P be that, at which all the angles APB, APC, BPC, are equal; * and from any other point Q, upon the lines forming them, let fall the three perpendiculars Qa, Qb, Qc. I say, first, that the sum of the three distances B



Aa, Bb, Cc, intercepted by those perpendiculars, and the three given points A, B, C, will be equal to the sum of the three first distances $AP + BP + CP$.

- For, if through the intersection M of Pc and bQ rv be drawn parallel to Aa, meeting Bb (produced in v, and PS, parallel to aQ, in S; it is evident (be
ⁱ Cor. to 3. cause the angle $v =$ ⁱ $BPa = vPM = \frac{2}{3}$ of a right-angle^m
^{1.} that the triangle Pvm is equilateral; and that the
^m Hyp.

* The determination of the position of a point, at which, line drawn from three given points, shall form any given angles, is given among the Geometrical Constructions, in the next section.

right-angled triangles QM_c , QM_r , having $QM_c (= Mb = vMb) = QM_r$, have also $cM = rM$; to which if $MP = Mv$ be added; so shall $cP = rv = aP + vS = P + Pb$.

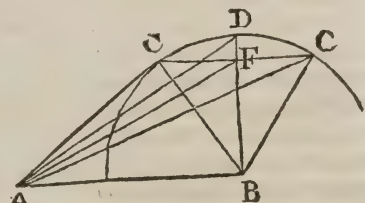
And, if to the first and last of these, $AP + BP + Cc$ be (again) added; then will $AP + BP + CP = Aa + b + Cc$, as was asserted.

Whence the Theorem itself is exceedingly obvious: for seeing that the sum $AP + BP + CP$ is but equal to the sum of the three bases Aa , Bb , Cc , it must necessarily be less than that of the three hypotenuses AQ , BQ and CQ . *Q. E. D.*

THEOREM IV.

The greatest triangle ABD that can possibly be contained under two right-lines, given in length, and any other right-line joining their extremes, will be when the two given lines AB, BD make right-angles with each other.

For, let BC be equal to BD , and the angle ABC either greater, or less than the right-angle ABD ; let also CF be drawn parallel to AB , meeting BD (produced, if



necessary) in F , and let A, F , and A, C be joined.

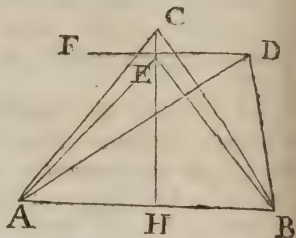
Then the angle BFC being a right-oneⁿ, it is evi- *5. 1.
 dent, that BC (BD) is $\perp BF^o$; and therefore the 20. 1.
 triangle ABD , being $\perp ABF^o$, is also greater than $\perp$ Ax. 2.
 its equal^o ABC . *Q. E. D.*

Cor. to
2. 2.

THEOREM V.

If all triangles ABC , ABD , having the same base AB , and the sum of their other sides the same, the isosceles one ACB , is the greatest.

Let CH be perpendicular to AB, and DEF parallel to AB, intersecting HC (produced if need be) in E; likewise let AE and BE be drawn.



^r 16. and

7. 1.

^e Theor. 1.

^t Hyp.

^u 23. 1.

^w Ax. 2. 1.

^x Cor. 1.

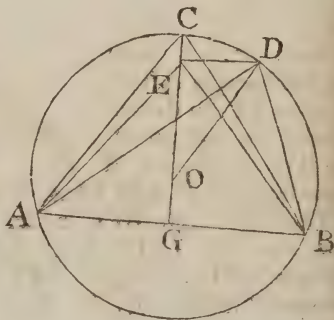
to 2. 2.

It is manifest, that the angles AEF, BED are equal^r; therefore $AE + BE \sqsubset AD + BD^s$; or than its equal $AC + BC^t$; and so the triangle AEB, falling within the triangle ACB^u, must be $\sqsubset ACB^w$; and therefore $ADB (=AEB^x) \sqsubset ACB$. Q. E. D.

THEOREM VI.

Of all triangles ABC, ABD standing upon the same base AB, and having equal vertical angles ACB, ADB, the isosceles one ACB is the greatest.

Let ACDB be a segment of a circle, in which the equal angles ACB, ADB are contained^y; make CEG perpendicular, and DE parallel, to AB; from the center O draw OD, and let A, E and B, E be joined. It is evident, that CG not only bisects AB^z, but also passes through the center O^a.



^y 22. 5.

and 11. 1.

^z 16. 1.

^a Cor. to 2.

3.

^e 20. 1.

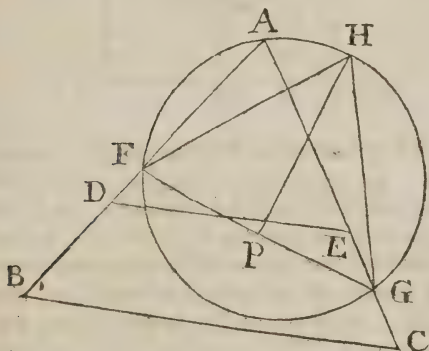
^b Cor. 2. 2.

and Ax. 2.

Therefore, OD (OC) being $\sqsubset OE^a$, the triangle ACB will also be $\sqsubset AEB^b$, or than its equal ADB^b. Q. E. D.

THEOREM VII.

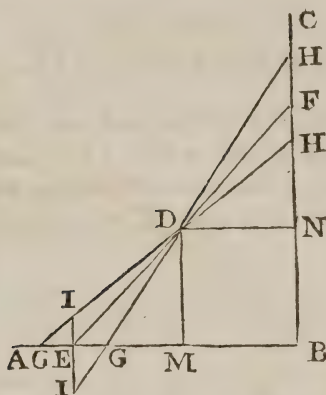
If all right-lines DE, FG that can be drawn to cut off equal areas ADE, AFG from a given triangle ABC, that DE is the least, which makes the triangle ADE, cut off, an isosceles one.



Let AFG be the circumference of a circle passing through the three points A, F, G; also let PH be perpendicular to FG, at the middle point P, meeting the circumference in H, and let FH and GH be drawn. The triangle FGH, being isosceles^c, is there-^c Ax. 10. fore $\square$ FAG^d, or than its equal^e ADE: Whence, as^d Theor. 6. the triangles FGH, ADE are equiangular^f, the base^e Hyp. FG of the greater, must consequently exceed the base^f Hyp. and Cor. to 10. DE of the less. Q. E. D. 1.

THEOREM VIII.

If all right-lines EF, GH, GH that can be drawn through a given point D, between two right-lines BA, BC given in position; that EF which is bisected by the given point D, forms with them the least triangle (EBF).

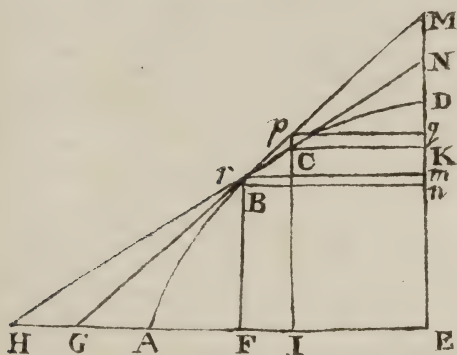


For, if EI, parallel to BC, be drawn, meeting GE^a in I; the^b equiangular triangles DFH and DEI having $DF = DE$ ^c, will be equal^k; and DFH will therefore be less, or greater than DEG^l, according as BG is greater or less than BE. In the former case let DEBH common, be added to both; so shall FEB be $\sqsubset$ HGB^k. And if, in the latter case, DGBF be added, then will HGB be $\sqsubset$ FEB^m; and consequently FEB (in this case also) $\sqsubset$ HGB. Q. E. D.

COROLLARY.

If DM and DN be drawn parallel to BC and BA the two equal^o triangles DEM, DFN, taken together (since $EM = DN$ ^o = MB ^p) will be equal to the parallelogram DMBN^q; and consequently to the parallelogram DMBN = $\frac{1}{2}$ BEF $\sqsubset$ $\frac{1}{2}$ BGH. Whence it is manifest, that a parallelogram is always less than half a triangle in which it is inscribed, except when the base of the one is half the base of the other in which case the parallelogram is just half the triangle.

SCHOLIUM.

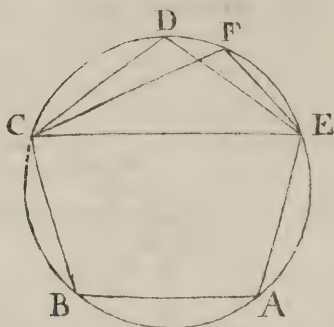


From the preceding Corollary alone, it may be very easily made to appear, that the least triangle HGM which can possibly be described about, and the greatest parallelogram EFBn that can be described in, any curve ABCD, concave to its axis AE, will be when the sub-tangent FG is equal to half the base EG of the triangle, or to the whole base EF of the parallelogram; and that the two figures will be in the ratio of *two to one*.

For let HN be a side of any other circumscribing triangle (EHN) touching the curve in C, and meeting Br in *r*: Then, the curve being concave to its axis, the point *r* will fall above B; whence, if *rm* be drawn parallel to Bn, then will $EGM = 2BE - 2rE - EHN$. Again, if IC, parallel to EM, be produced to meet M in *p*, and CK and *pq* be drawn parallel to AE; then, also, will $BE = \frac{1}{2}EGM - pE - CE$, as was to be shown.

THEOREM IX.

If all right-lined figures, contained under the same number of sides, and inscribed in the same circle, that is the greatest whose sides are all equal.

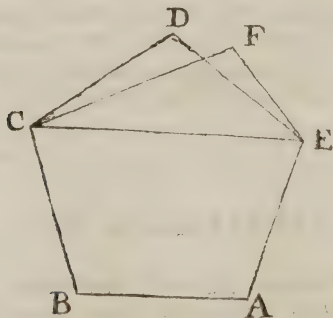


For, *if possible*, let some polygon ABCFE whose sides CF, EF are unequal, be the greatest.

Let CDE be an isosceles triangle described in the same segment with CFE; which being $\square$ CFE, the whole polygon ABCDE will also be $\square$ the whole polygon ABCFE; *which is repugnant*. Therefore the polygon is the greatest when the sides are all equal. Q. E. D.

THEOREM X.

Of all right-lined figures, contained under the same perimeter, and number of sides, the greatest is, when the sides are all equal.

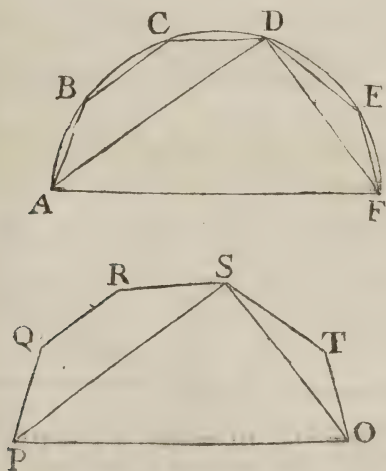


For, if ABCDE be the greatest possible, the triangle CDE must, manifestly, be $\square$ any other triangle

(FE upon the same base, of which the sum of the other sides is also the same. But, (*by Theorem V.*) the greatest triangle, when the base, and the sum of the sides are given, is that whose sides are equal: Therefore DC and ED are equal. In the same manner it appears, that $BC=CD$, &c. Q. E. D.

THEOREM XI.

If all the sides of a polygon, except one, be given in length, and their position be required, so as to make the polygon itself the greatest possible; I say, their position must be such, that two lines drawn from the extremes of the unknown side to any angle of the polygon, shall form a right angle.



For, if you would have the polygon ABCDEF to be the greatest possible, and yet ADF, subtended by the unknown side AF not a right-angle: Then let $\angle PSO$ be a right-angle, contained under $PS=AD$, and

$OS=FD$; and upon PS and OS , let the figures $PSRQ$ and OST be described, equilateral, and equal, to

* 10. 6. $ADCB$ and FDE ¹.

¹ Theor. 4. The triangle $PSO \sqsubset ADF$ ²; therefore, $PSRQ$

² Constr. being $= ADCB$, and $OST = FDE$ ³, the whole polygon

³ Ax. 6. 1. $PQRSTO$ is also $\sqsubset$ the whole polygon $ABCDEF$ ⁴ which is repugnant.

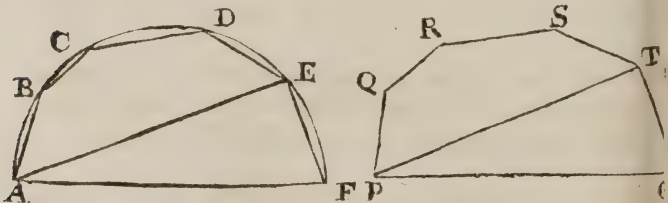
COROLLARY.

* 13. 3. Hence, because the angle in a semi-circle is a right angle ⁵; it appears, that the *greatest polygon that can be contained under any proposed number of given sides, and one other side any how taken, will be when it may be inscribed in a semicircle, of which the indetermined line will be the diameter.*

THEOREM XII.

A polygon $ABCDEA$ in a circle, is greater than any other polygon $PQRSTP$, whatever, whose sides are the same both in length and number.

Let AF be the diameter of the circle, and join AE and EF ; also make the angle $PTO = AEF$, $TO = EF$, and let PO be drawn.



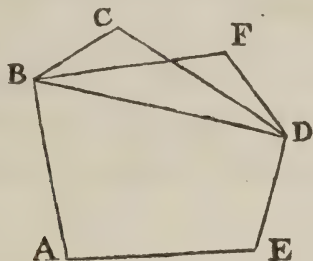
* Hyp. Because $AB=PQ$, $BC=QR$, $CD=RS$, $DE=ST$, and $EF=TO$ ⁶, the polygon $ABCDEF$, being inscribed in a semi-circle, will be $\sqsubset$ the polygon $PQRSTO$ ⁷; and, if from these, the equal triangles AEF , PTO be taken away there will remain $ABCDEA \sqsubset PQRSTP$. Q. E. D.

* Hyp.

* Theor. 11.

SCHOLIUM.

That the magnitude of the greatest polygon, which can be contained under any number of unequal sides, does not at all depend upon the order in which those



lines are connected to each other, will appear, thus, Let ABCDE be the greatest, one way, or according to one order of the sides; and upon BD let a triangle BDF be constituted whose sides DF and BF are, respectively, equal to BC and DC; then, the triangles BCD, BFD being equal, the whole polygons ABCDE and ABFDE will likewise be equal, notwithstanding their equal sides BC, DF, &c. are placed according to different orders.

THEOREM XIII.

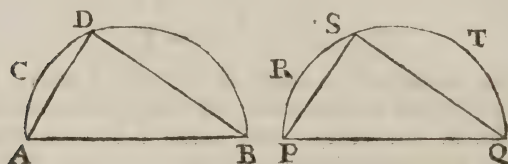
Of all polygons, contained under the same perimeter, and number of sides; that whose sides, and angles, are equal, is the greatest.

For, the greatest polygon that can be contained under a given perimeter, is one whose sides are all equal^a. But of all the polygons of this sort, that is the greatest which may be inscribed in a circle^b: Therefore, the greatest of all, is that whose sides are all equal, and which may be inscribed in a circle, or whose angles, as well as the sides, are all equal. Q. E. D.

^a Theor.
10.
^b Theor.
12.

THEOREM XIV.

The greatest area that can possibly be contained by one right-line, any how taken, and any other line or lines, whatever, of which the sum is given: will be, when two right-lines drawn from the extremes of the unknown line first mentioned, to meet any where in the given boundary, make right-angles with each other.



For, if you would have the area ACDEBA, contained by some right-line AB, and ACDEB whose length is given, to be the greatest possible, and ADB, at the same time, not a right-angle: Then, let PSQ be a right-angle, contained under $PS=AD$, and $QS=BD$; and, having joined PQ, upon PS and QS conceive two figures PRS and QST to be formed equal and alike in all respects to ACD and DBE.

^c Theor. 4.

Since the area $PSQ \sqsubset ADB^c$; it is manifest, that the area PRSTQP, contained by the right-line (PQ and PRSTQ ($=ACDEB^d$) will also be $\sqsubset$ the area

^d Hyp.

^e Ax. 6.

ACDEBA^e; which is repugnant.

Therefore the area ACDEBA cannot be the greatest possible, unless the angle ADB be a right one. Q. E. L

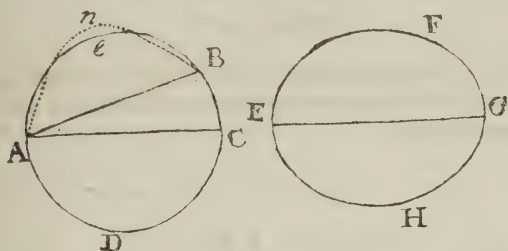
COROLLARY.

^f 13. 3

Hence, because the angle in a semi-circle is a right angle^f, it is evident, that *the area will be the greatest possible, when the given length or boundary, form the arch of a semi-circle; of which the indeterminate right-line proposed is the diameter.*

THEOREM XV.

If all plane figures ABCD, EFGH, contained under equal perimeters (or limits), the circle (ABCD) is the greatest.



For, if the diameter AC be drawn, and EFG be taken equal to the arch ABC; then the area ABCA will (by the precedent) be $\sqsubset$ the area EFGH, contained by EFG and the right-line EG; and ADCA will also be $\sqsubset$ EHGE: Therefore ABCD $\sqsubset$ EFGH. Q. E. D.

COROLLARY.

Hence it appears, that *the greatest area that can possibly be contained by a right-line AB, and a curve-line AeB, both given in length; will be, when the latter is an arch of a circle.*

For, let AnB be any other curve line, equal to AeB, and let the whole circle AeBCD be completed; which will (it is proved) be greater than the mixed figure AnBCD; and consequently, by taking away the common segment ABCD, there will remain AeBA $\sqsubset$ AnBA.

THEOREM XVI.

The greatest parallelepipedon that can be contained under the three parts of a given line AB, any how taken, will be when all the parts are equal to each other.

- For, if possible, let two parts AE, ED be $A \xrightarrow{\quad C \quad E \quad D \quad} B$ unequal. Bisect AD in C; then will the rectangle under AE ($AC + CE$) and ED ($AC - CE$) be $\sqsupset AC^2$ (or $AC \times CD$) by the square of CE^g . Therefore the solid $AE \times ED \times DB$ will also be $\sqsupset$ the solid $AC \times CD \times DB^h$; which is contrary to hypothesis.
- ^g 7. 2.
- ^h 21. 7.

COROLLARY.

Hence, of all rectangular parallelepipeds having the sum of their three dimensions the same, the cube is the greatest.

THEOREM XVII

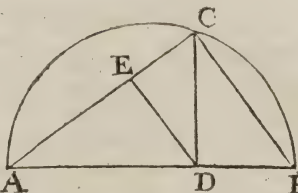
The greatest parallelepipedon $AC^2 \times CB$ that can possibly be contained under the square of one part AC, of a given line AB, and the other part BC, any how taken; will be, when the former part is the double of the latter.

- For, let Ac and Bc be any other $A \xrightarrow{\quad d \quad D \quad c \quad C \quad} B$ parts into which the given line AB may be divided; and let AC and Ac be bisected in D and d. So shall
- ⁱ Cor. to 6. $AC^2 \times CB = 4AD \times DC \times CB^i \sqsupset 4Ad \times dc \times CB^k (AC^2$
- ^{2.} and 7. $\times CB^i)$ by the precedent. Q. E. D.
- ^k Cor. to 6. 2.

THEOREM XVIII.

The hypotenuse AB of a right-angled triangle ABC being given; the solid $BC \times AC^2$ contained under one leg BC and the square of the other AC, will be the greatest possible, when the square of the latter leg AC is double to that of the former BC.

For, if CD be conceived perpendicular to AB , and DE to AC ; it will be $AC^2 (AB \times AD^1)$: $AB^2 :: AD : AB^m :: DE : BC^n$; and consequently, $AC^2 \times BC = AB^2 \times DE^o$; A D B



¹ Cor. to 19. 4.
^m 21. 7.
ⁿ 14. 4.
^o 23. 7.

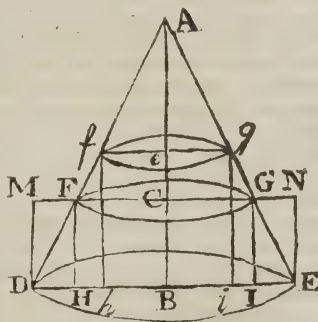
which (as AB^2 is given) will, evidently, be the greatest possible, when DE , or its square^p (DE^2) is the greatest possible. But $DE^2 : AD^2 :: BC^2 (BD \times AB^1) : AB^2 :: BD : AB^m$; and therefore $DE^2 \times AB = AD^2 \times BD^o$; which (and consequently DE^2) will be the greatest possible, when AD is the double of BD^r ; that is, when $AC^2 (AD \times AB)$ is the double of $BC^2 (BD \times AB)$. *Q. E. D.*

^p Cor. 2. to 6. 2.
^q Cor. to 11. 4.
^r Theor. 17.

THEOREM XIX.

The altitude BC of the greatest cylinder HG that can possibly be inscribed in any cone ADE , is one third part of the altitude AB of the cone, and the cylinder itself $\frac{4}{9}$ parts of the cone.

For, let gh be any other cylinder inscribed in the cone; and it will be, $AC^2 \times BC : CG^2 \times BC :: AC^2 : CG^2 :: AC^2 : cg^2 :: AC^2 \times Bc : cg^2 \times Bc$; whence, by alternation, $AC^2 \times BC : AC^2 \times Bc :: CG^2 \times BC : cg^2 \times Bc$; and so likewise is the cylinder HG to the cylinder^u hg ; but $AC^2 \times BC \sqsubset^w AC^2 \times Bc$; therefore $HG \sqsubset hg$. Again, since $AC = \frac{2}{3} AB^x$, and therefore $CG = \frac{2}{3} BE$; we also have, cylinder HG : cone ADE (or cylinder^v DN) :: $CG^2 (\frac{4}{9} BE^2) : BE^2 :: \frac{4}{9} : 1$. *Q. E. D.*



²² 7.
^{Cor.} 11. 4.
³ and 5.
^{of} 8.
^{Theor.} 17.
^{ax.} 4.
^{Cor.} 3. to 8. 8.
²¹ 7.
and 3. 3.

SCHOLIUM.

From this proposition, by reasoning as in the Scholium to Theorem VIII, it will appear, that *the least cone that can be described about, and the greatest cylinder that can possibly be described in, any solid generated by the rotation of a curve, concave to its axis, will be, when the sub-tangent is two-thirds of the altitude of the cone, or twice the altitude of the cylinder; and that the two figures will be in the ratio of nine to four.* From whence the dimensions of the greatest and least cylinders and cones, that can be described in, and about solids generated by curves, to which the method of drawing tangents is known, may be readily determined."

"The first ideas of Maxima and Minima are to be found in the Elements of Euclid, or flow immediately from them: thus it appears by Prop. 5, B. II, that the greatest rectangle that can be made of the two parts of a given line, any how divided, is when the line is divided equally in the middle: Prop. 7, B. III, shows that the greatest line that can be drawn from a given point within a circle, to its circumference, is that which passes through the center; and, that the least line that can be so drawn, is the continuation of the same to the other side of the circle:—Prop. 8, *ib* shows the same for lines drawn from a point without the circle, and many more instances of a similar nature might be pointed out in the Elements. Other writers on the Maxima and Minima are *Apollonius*, in the whole 5th Book of his *Conic Sections*:—*Archimedes*, in Prop. 9. of his treatise on the *Sphere and Cylinder* where he demonstrates, that of all spherical segments under equal superficies, the hemisphere is the greatest: *Serenus*, in his 2nd Book, or that on the *Conic Sections*:—*Pappus*, in many parts of his *Mathematical Collections*; as in Book III, Prop. 28, &c., B. VI Prop. 31, &c., where he treats of some curious cases of variable geometrical quantities; showing how some increase and decrease both ways, to infinity, while others proceed only one way, by increase or decrease to infinity, and the other way to a certain magnitude, giving a Maximum and a Minimum; also B. VII, Props. 13, 14, 165, 166, &c. These are the principal writers on the geometrical Maxima and Minima among the ancients, to which may be added some others of a more recent date, as *Richard White* in his *Hemisphaerium Dissectum*, 4to. Rom. 1647. *Viviani de Maximis et Minimis*, &c. Fol. Flor. 1659.—*Simon Lhuillier* c Geneva; *Bossut*; *Le-Gendre*; *Dr. Hutton*; *Dr. Olinthus Gregory*, &c. &c."

THE
CONSTRUCTION
OF

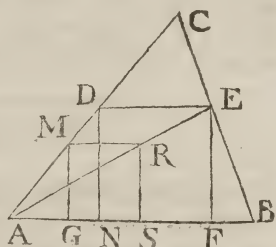
GEOMETRICAL PROBLEMS.

PROBLEM I.

*In a given triangle ABC, to inscribe a square DEFN.**

CONSTRUCTION.

FROM any point M, in either side, upon the base AB, let fall the perpendicular MG; make MR perpendicular, and equal, to it, and let ARE be drawn, meeting the other side of the triangle in E; then draw ED parallel, and EF and DN perpendicular, to AB; and the thing is done.



DEMONSTRATION.

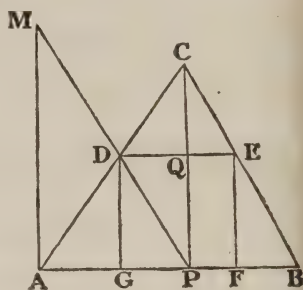
Let RS be drawn parallel to EF: Then (*by similar triangles*) $RS (MG) : EF :: AR : AE :: MR : DE$: Therefore, as MG and MR are equal, *by construction*, EF and DE will likewise be equal.

By the same method a rectangle may be inscribed in a triangle, whose sides shall be in a given ratio; if MR and MG (instead of being equal) be taken in the given ratio; the rest of the construction being exactly the same.

* D'Omerique, *Anal. Geom.* B. II, Prop. 21.

The same constructed otherwise.

Upon the base AB, from the vertex C, let fall the perpendicular CP, and make AM parallel to CP, and equal to AB; also draw PM, cutting the side AC in D, and draw DE parallel to AB, meeting BC in E; then upon AB let fall the perpendiculars DG, EF, and the thing is done.



DEMONSTRATION.

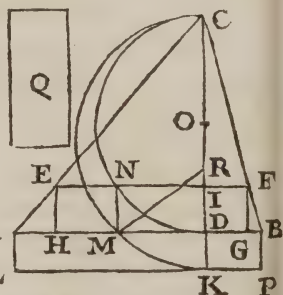
It is evident, from the construction, that the figure DEFG, is a rectangle. And, by similar triangles $AP : AM (AB) :: PG (QD) : DG$; also (by 20. 4.) $AP : AB :: DQ : DE$; therefore $DG = DE$, and consequently DEFG is a square. Q. E. D.

PROBLEM II.

In a given triangle ABC, to inscribe a rectangle EFGH, equal to any given right-lined figure Q, not exceeding half the triangle.

CONSTRUCTION.

On the base AB (by 7, 6.) let a rectangle ABPL be constituted = Q; and let LP meet the perpendicular CD of the triangle (produced) in K. Then (by 17. 5.) let CD be divided in I, so that $CI \times DI = CD \times DK$ (that is, let two semi-circles be described on CD and CK; drawing MN and NI parallel to CD and AB): So shall DI be the altitude of the required rectangle.



DEMONSTRATION.

Since (by *Constr.*) $CI \times DI (= NI^2 = MD^2) = CD \times DK$, thence will $DI : DK :: CD : CI :: AB : EF$ (by 20. 5.); and consequently $DI \times EF$ (by 10, 4.) = $AB \times DK = Q$. *Q. E. D.*

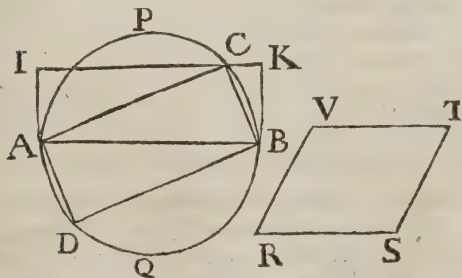
Limitation. That the Problem will be impossible, when Q is greater than half the triangle, is evident from the Construction, as well as from the Theorem on *p.* 192. It may also be observed, that there is another way, besides that used above, for dividing CD in the manner proposed; which (though not more obvious) is in point of conciseness, rather preferable; and is thus.

Having (as before) found a mean proportional DM between CD and DK , and bisected CD in O ; from M to CD apply $MR = OD$, and take $OI = RD$. So shall $CI \times DI = OD^2 - OI^2$ (by 7. 2.) = $MR^2 - RD^2$ (by *Hyp.*) = $DM = CD \times DK$ (as before).

PROBLEM III.

In a given circle $APBQ$, to describe a rectangle equal to a given right-lined figure, $RSTU$, not exceeding half the square of the diameter.

CONSTRUCTION.

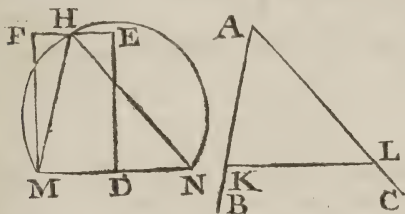


Upon the diameter AB describe the rectangle $ABKI = RSTU$ (by 7. 6.); and from the point C ,

PROBLEM V.

Between two lines AB, AC , given by position, to apply a line KL , equal to a given line MN , so that the triangle AKL formed from thence, shall be of a given magnitude.*

CONSTRUCTION.



Having bisected MN in D on MD describe a rectangle $MDEF$ (by 7. 6.) = the magnitude given: also on MN let a segment of a circle be described (by 22. 1.) to contain an angle $= A$; and from its intersection with EF , draw HM and HN ; then make $AK = HM$, $AL = HN$, and join K, L . So shall the base KL be also = the base MN , and the triangle AKL equal and like in all respects, to HMN ; which last (by Cor. to 2. 2.) is manifestly, equal to the magnitude given $MDEF$. Q. E. D.

PROBLEM VI.

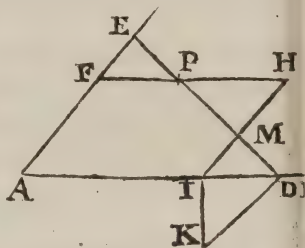
Through a given point P , to draw a line EPD to meet two lines AB, AC , given by position, so that the triangle ADE formed from thence, shall be of a given magnitude.†

* D'Omerique, *Anal. Geom.* III, 36. Saunderson's *Algebra*, pt. 309. Appendix to the Author's *Algebra*, prob. 33.

† Newton's *Arith. Universalis*. Prob. 20;—Leslie's *Geometry*, page 305.

CONSTRUCTION.

Draw FPH parallel to AB , intersecting AC in F ; and on AF , let a parallelogram $AFHI$ be formed, equal to the given area of the triangle: Make IK perpendicular to AI , and equal to FP ; and from K , to AB , apply $KD=PH$; then draw DPE , and the thing done.



DEMONSTRATION.

The triangles PHM , PFE and MDI , by reason of the parallel lines, are similar; and therefore, since the three homologous sides PH (KD), FP (IK) and D are such, as to form a right-angled triangle (by Constr. the triangle PHM on the first of them, is equal to both the other two PFE and MDI (by 29. 4.): and, if these equal quantities, $AFPMI$ be added; then will $AFHI$ be also ADE .

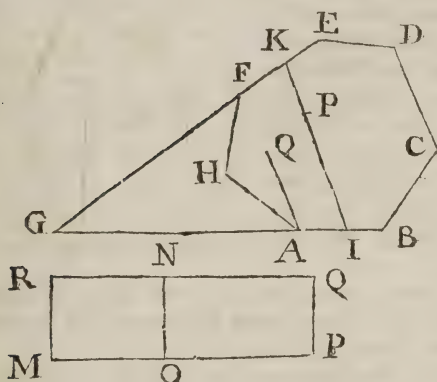
Limitation. This problem will be impossible, when KD (PH) is less than KI (PF); that is, when the area given is less than a parallelogram under AF and $2FP$.

PROBLEM VII.

From a given polygon $ABCDEFH$, to cut off part $AIKFH$, equal to a given rectangle MN by a line (IK), either parallel to a given line AQ , or passing through a given point P .*

* Newton, *Arith. Universalis*, Cor. to Prob. 20.

CONSTRUCTION.



Let BA, and EF be produced to meet in G; and upon ON let a rectangle OQ be constituted (by 7. 6.) equal to AGFH; then, by Prob. the 4th, or 6th, according to the case proposed, draw IK, so as to make the triangle GKI = MQ, and the thing is done.

The Demonstration is manifest from the Construction.

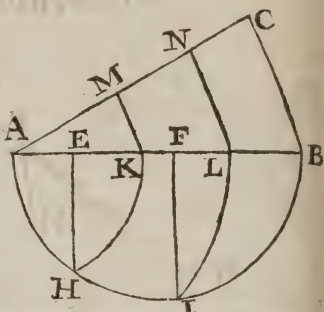
And, in the same manner, the polygon may be divided according to any given ratio; because, the whole being given, each part will be given.

PROBLEM VIII.

To divide a given triangle ABC into any proposed number of parts (AKM, KN, LC) so as to have any given proportion to each other; by means of lines drawn parallel to one of the sides BC of the triangle.

THE CONSTRUCTION OF CONSTRUCTION.

Let AB be divided into parts, AE , EF , FB , having the same given proportion to each other, as the parts of the triangle are to have. Upon AB let a semi-circle $AHIB$ be described; and perpendicular to AB , draw EH , FI , meeting the circumference in H and I : From the center A , through H and I , describe the arcs HK , IL , meeting AB in K and L ; then draw KM and LN parallel to BC , and the thing is done.



DEMONSTRATION.

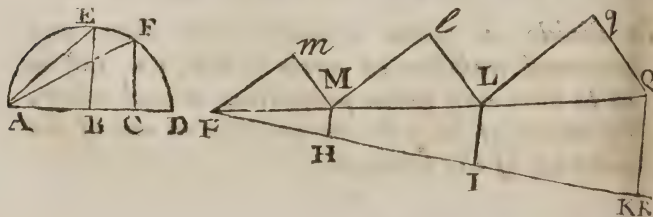
The triangles AKM , ALN and ABC , are in proportion, to one another, as AK^2 ($AB \times AE$), AL^2 ($AB \times AF$) and AB^2 (by 19. and 24. 4.); that is, as AE , AF , and AB (by 7. 4.) Whence (by division) the proposition is manifest.

PROBLEM IX.

To divide a given line PQ into any proposed number of parts, so that similar right-lined figures Pmm , MLl , LQq , described upon them, shall have the same given ratio among themselves, as an equal number of right-lines AB , AC , AD assigned.

CONSTRUCTION.

Upon the greatest AD of the given lines AD , AC ,



AB, describe a semi-circle AEFD; and perpendicular to AD, draw BE and CF, meeting the circumference in E and F; and, having drawn PR, at pleasure, in it take $PH = \text{dist. AE}$, $HI = \text{dist. AF}$, and $IK = AD$; draw KQ, and parallel thereto draw HM, L; which will divide PQ, as required.

DEMONSTRATION.

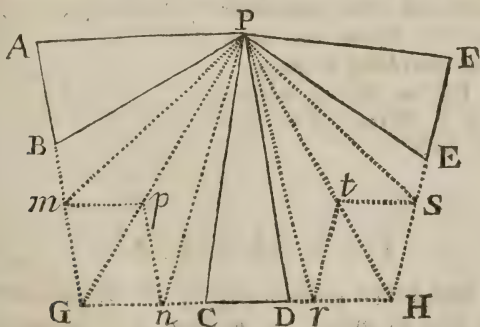
$PMm : LQq :: PM^2 : LQ^2$ (by 26. 4.) $:: PH^2 (= AE^2 = AB \times AD, \text{ by 19. 4.}) : IK^2 (AD^2) :: AB : AD$ (by 7. 6.) In the very same manner it appears, that $MLl : LQq :: AC : AD$. Q. E. D.

PROBLEM X.

To determine the position of a point P, so that lines drawn from thence to the extremes of three right-lines AB, CD, EF, given in length and position, shall form three triangles $\triangle ABP$, $\triangle CPD$, $\triangle EPF$, mutually equal to each other.

CONSTRUCTION.

Let the given lines be produced to meet in G and H; in which take $Gm = AB$, $Hs = EF$, and Gn , Hr ,



equal each to CD: Complete the parallelograms mpn , $Hrts$; and let the diagonals Gp and Ht be produced till they meet in P, and the thing is done.

DEMONSTRATION.

Let PA, PB, Pm, &c. be drawn. The triangles GPn, GPm, having the same base GP, and equal altitudes (because GMpn is a parallelogram) are therefore equal to each other: But CPD = GPn, and APB = GPm, (*by Cor. to 2. 2.*); whence CPD and APB are likewise equal. And, from the very same reasoning, it appears, that CPD and FPE are equal. Q. E. D.

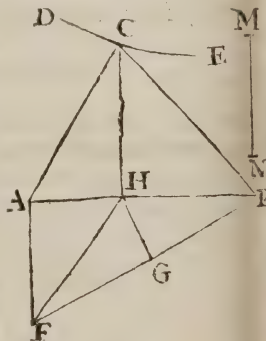
PROBLEM XI.

From two given points A, B, to draw two lines AC, BC, to meet in a line DE of any kind, given by position: so that the difference of their squares shall be equal to a square given (MN^2).

CONSTRUCTION.

Make AF perpendicular to the line AB, and equal to MN; draw BF, which bisect with the perpendicular GH; and from its intersection with AB, draw HC perpendicular to AB, meeting DE in C; draw AC and BC, and the thing is done.

For $BC^2 - AC^2 = BH^2 - AH^2$ (*by 9. 2.*) $= FH^2 - AH^2 = AF^2 = MN^2$.

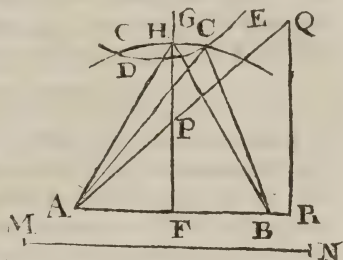


PROBLEM XII.

From two given points A, B, to draw two lines AC, BC, to meet in a line DE of any kind, given by position; so that the sum of the squares shall be equal to a given square, MN^2 .

CONSTRUCTION.

Bisect AB with the perpendicular FG , in which take $FP=FA$; and draw $APQ=MN$; on AB (produced, if necessary) let fall the perpendicular QR ; from A to FG apply $AH=AR$, and about the center F , through H , let the circumference of a circle be described; and from its intersection C with the given line DE , draw CA and CB , and the thing is done.



DEMONSTRATION.

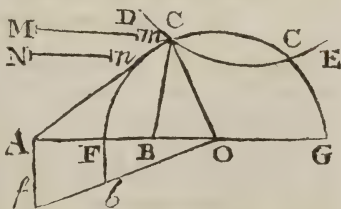
Let BH be drawn; which being $=AH=AR=RQ$ (because $FP=AF$); thence will $AC^2 + BC^2 = AH^2 + BH^2$ (by 20. 3.) $= AR^2 + RQ^2 = AQ^2 = MN^2$. *Q. E. D.*

PROBLEM XIII.

From two given points A, B , to draw two lines AC, BC , meeting in a line DE of any kind, given by position; so as to obtain the ratio of two unequal right-lines Mm, Nn assigned.*

CONSTRUCTION.

Having joined the given points, divide AB in F (by 15. 5.) so that $AF : BF :: Mm : Nn$; make Af and Fb parallel to each other, taking the former $= AF$, and the latter $= FB$; and through



* See Pappus, *Math. Coll.* VII, 155;—D'Omerique, *Anal. Geom.* II, 35;—the Author's *Select Exercises*, Prob. 19, the Appendix to his *Algebra*, Lemma, page 334;—and Theo. B, on page 84 of this volume.

their extremes draw fbO , meeting AB , produced, in O ; from whence, with the radius OF , let the semi-circle FCG be described, cutting DE in C ; then draw AC and BC , and the thing is done.

DEMONSTRATION.

Because $OA : Af (AF) :: OF : Fb (FB)$, we have (by division) $OA : OF :: OF : OB$; or $OA : OC :: OC : OB$. And so, the triangles OAC , OCB , having one angle FOC common, and the sides about it proportional, must, therefore, be similar (by 15. 4); whence the other sides will also be proportional, or $AC : BC :: OA : OC (OF) :: Af : Fb :: Mm : Nn$ (by Constr.) *Q. E. D.*

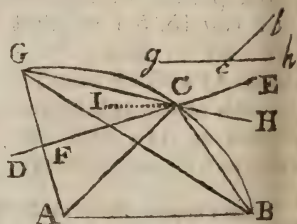
Note, When, in either of the two preceding problems, the circle described, neither cuts nor touches the given line DE , the thing proposed to be done, will be impossible; as no two lines drawn from A and B , to meet above the circumference, can possibly have their ratio, or the sum of their squares, the same as two lines meeting in the circumference.

PROBLEM XIV.

From two given points A , B , to draw two lines AC , BC , meeting in a right-line DE , given by position; so as to make therewith two angles ACD , BCE , whose difference shall be equal to an angle given, bch .

CONSTRUCTION.

Make AFG perpendicular to DE , and $FG = AF$; and, having drawn GB , on it let a segment of a circle GCB be described (by 22. 5.) to contain an angle equal to the supplement (bcg) of the given one bch ; and from its intersection (C) with DE , draw CA and CB ; and the thing is done.



For, if BC and GCH be drawn, then will $ACD = GCD = ECH = BCE - BCH$ (*bch*).

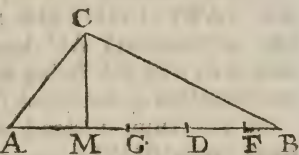
In the same manner, the Problem will be constructed, when, instead of the difference of ACD and BCE , that of ABC and BAC is given: Because, when DE is parallel to AB , the latter difference is equal to the former; and, in all other cases, differs from it by twice the given angle GCI , expressing the inclination of the said lines.—When the sum of the angles ACD and BCE is given, the angle ACB is also given: And here, nothing more is necessary, than barely to describe, upon AB , a segment of a circle to contain the said given angle ACB .

LEMMA.

If, of any three proportional lines, AB , DB , FB , the difference AF of the two extremes be bisected in G ; and if on the greatest AB , as a base, a triangle, ABC be so formed, that its less side AC shall be to the distance MG of the perpendicular from the bisecting point G , in the given ratio of AB to DB ; then shall the greater side BC exceed the less AC by the given line DB .

DEMONSTRATION.

Because FM exceeds AM by $2GM$, BM will exceed it by $2GM + BF$; and the rectangle under this excess and the whole base AB ($= 2MG \times AB + BF \times AB$) will therefore (*by* 9. 2.) be $= BC^2 - AC^2$. But (*by hypothesis and* 10. 4.) $MG \times AB = AC \times BD$ and $BF \times AB = BD^2$: Therefore $2AC \times BD + BD^2 = BC^2 - AC^2$; and, by adding AC^2 , common, $AC^2 + 2AC \times BD + BD^2$ (or the square of $AC + BD$, *by* 6. 2.) will be $= BC^2$; and, consequently, $AC + BD = BC$.
Q. E. D.



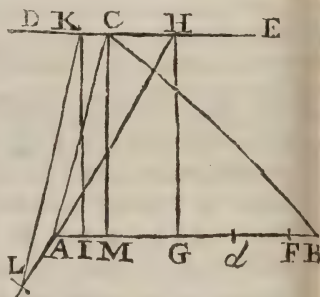
This *Lemma* is not only of use in the Problem next following, but will be found a ready instrument in the solution of many others; for which reason it is here put down.

PROBLEM XV.

*From two given points A, B, to draw two lines AC, BC to meet in a right-line DE, given by position; so that their difference shall be equal to a given right-line Bd.**

CONSTRUCTION.

In AB, take a third-proportional BF to BA and Bd; and, having bisected AF in G, take GI = Bd; make GH and IK perpendicular to AB, meeting DE in H and K; and draw HAL, to which from K, apply KL = AB; and parallel thereto draw AC, meeting DE in C; join B, C; *and the thing is done.*



DEMONSTRATION.

Let CM be perpendicular to AB.

Then, because of the parallel lines, it will be $AC : KL (AB) :: HC : HK :: GM : GI (Bd.)$ Whence (*by alternation*) AC being to GM , as BA to Bd , and AB, dB, FB being also proportionals; the whole construction is manifest *from the Lemma* premised.

If the sum, instead of the difference, of the two lines (AC, BC) be given, the method of construction will be exactly the same, without the least alteration of any one step; provided that B*d* be first of all taken (in BA, produced) equal to the given sum, instead of the difference.

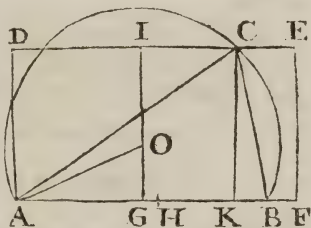
* Gregory St. Vincent, *de Quad. Circ.* prop. 82;—D'Omerique, *Anal. Geom.* III, 26.

PROBLEM XVI.

From two given points, A, B , to draw two lines AC, BC , so as to meet in a right-line DE parallel to that (AB) joining the said points, and that the rectangle $(AC \times BC)$ contained by them, shall be equal to a rectangle given, $ADEF^*$.

CONSTRUCTION.

Bisect AB and AF , in G and H ; and, having drawn GI perpendicular to AB , to it, from A , apply $AO=AH$; and from the center O , through A and B , let the circumference of a circle be described intersecting DE in C ; from whence draw CA and CB , and the thing is done.



DEMONSTRATION.

For, if CK be drawn perpendicular to AB , it is evident (by 25. 3.) that $AC \times BC = CK \times 2AO = AD \times AF$. $Q. E. D.$

PROBLEM XVII.

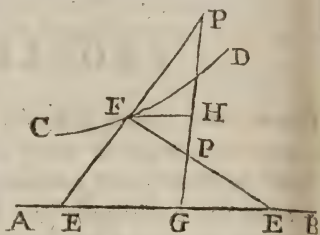
Through a given point P , so to draw a line FPE that the parts PF, PE , intercepted by that point and two lines AB, CD , given in position, shall obtain a given ratio.†

* D'Omerique, *Anal. Geom.* I, 31. *Simp. Exer. Prob.* 21.

† Apollonius, *de Sectione, Rationis*;—Newton's *Principia*, B. II, Lemma 7.

THE CONSTRUCTION OF CONSTRUCTION.

Through P , to any point in AB , draw PG , in which (*by* 13. 5.) take PH to PG , in the given ratio of PF to PE ; draw HF parallel to AB , meeting CD in F ; then draw FPE , and the thing is done.



For, the triangles PGE , PFH being equiangular (*by* 3. and 7. 2.) thence is $PF : PE :: PH : PG$: and so PF and PE (as well as PH and PG) are in the ratio given.

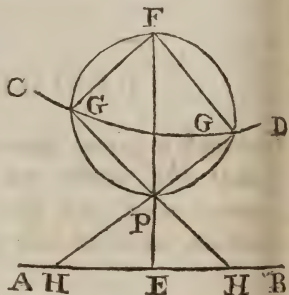
In this construction, it is necessary, that one of the two given lines should be a *right-one*: The other may, it is manifest, be a line of *any kind* whatever.

PROBLEM XVIII.

Through a given point P , to draw a line GH terminating in a right-line AB , and in a line CD of any kind, given both by position; so that the rectangle under the two parts PG , PH , shall be of a given magnitude.*

CONSTRUCTION.

Having drawn EPF perpendicular to AB , take therein PF (*by* 7. 6.) so that $PE \times PF =$ the magnitude given: Upon PF , as a diameter, let a circle be described, intersecting CD in G : then draw GPH , and the thing is done.



* A Problem of Apollonius, *de Sectione Spatii*. See Halley's *Restitution* of that work, 8vo. Oxon. 1707.

DEMONSTRATION.

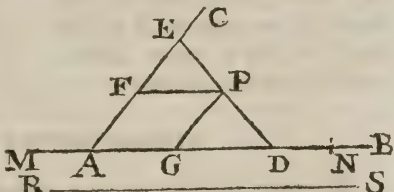
If FG be drawn, the triangles PFG and PHE, having $FPG = HPE$, and $FGP (= \text{a right-angle, by 13. 3.}) = PEH$, will be equiangular: And, consequently (by 24. 3.) $PG \times PH = PF \times PE = \text{the magnitude given. Q. E. D.}$

PROBLEM XIX.

Through a given point P, between two right-lines AB, AC, given by position, so to draw a line ED, that the sum of the segments ($AD + AE$) cut off by it, from the two former, shall be equal to a given line RS.*

CONSTRUCTION.

Draw PF and PG parallel to AB and AC; in BA produced take $AM = AF$, and $MN = RS$; then divide GN in D (by 17. 5.) so that $GD \times ND = AM \times AG$; draw DPE, and the thing is done.



DEMONSTRATION.

By similar triangles, $GD : GP (= AF = AM) :: FP (= AG) : FE$; and consequently $GD \times FE = AM \times AG = GD \times ND$ (by Constr.): Whence $FE = ND$; and therefore $FE + AF + AD (AE + AD) = ND + AM + AD = MN = RS$. Q. E. D.

Limitation. It appears from the construction, that the problem will be impossible, when (GM) the excess of RS above PF and PG, is less than the double of a mean-proportional between these two quantities.

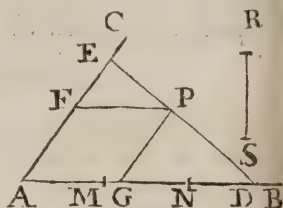
* Select Exercises, prob. 43.

PROBLEM XX.

Through a given point P , between two right-lines AB , AC , given by position, so to draw a line DE , that the difference ($AD - AE$) of the segments cut off by it from the two former, shall be equal to a given line RS .

CONSTRUCTION.

Draw PF and PG parallel to AB and AC ; in AB take $AM = AF$, and $MN = RS$; then (by 18. 5.) let a line ND be added to GN , so that $GD \times ND = PG \times PF$; draw DPE , and the thing is done.



DEMONSTRATION.

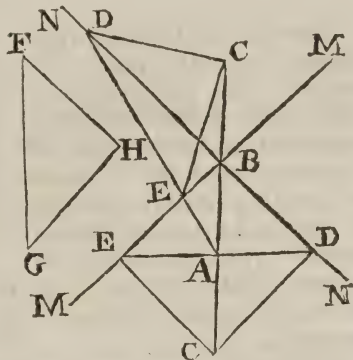
Because of the similar triangles PGD , EFP , we have $GD \times FE = PG \times PF = GD \times ND$ (by Constr.) and consequently $FE = ND$; whence $AD - AE = AN - AF$ (AM) $= MN = RS$. *Q. E. D.*

PROBLEM XXI.

Between two right-lines NBN , MBM , given by position, to apply a line DE of a given length, and which (produced, if necessary,) shall pass through a given point A , equally distant from the said given lines.*

* *Algebra*, Appendix, prob. 72.

Let FG be the length given : and, having drawn AB, make the angles F, G equal each to ABD (or ABE); and in AB produced take AC (*by* 18. 5.) so that $AC \times BC = HG^2$: from C to NN apply $CD = HG$: then draw DAE (DEA), and the thing is done.

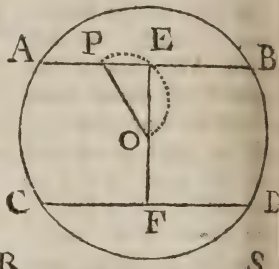


Because $AC : CD :: CD : BC$ (*by Constr. and 10.*
b.) the triangles CAD and CDB (having one angle
common, and the sides about it proportional) will be
equiangular (*by 15. 4.*): And therefore, since $CDA =$
 $DBC = ABE$ (*by hypothesis*), the circumference of a
circle may be described through all the four points,
C, D, B, E (*by 11. 3. or 19. 3.*) And so the angle
DEC, standing on the same subtense (DC) with
DBC, will be equal to it; and, consequently, equal to
EDC. Therefore the isosceles triangles EDC, FGH
being equiangular, and having $CD = HG$, their bases
ED and FG will also be equal. *Q. E. D.*

In a given circle $ABDC$, to apply a chord AB equal to a given line RS (less than the diameter) and which shall pass through a given point P .

CONSTRUCTION.

Inscribe $CD=RS$, upon which, from the center O , let fall the perpendicular OF ; also draw OP , upon which let a semi-circle be described, and in it apply $OE=OF$; lastly, through P and E , let AB be drawn; which will, manifestly be equal to CD ($=RS$, by 3. 3.) R ————— S as being equally distant from the center, by construction.



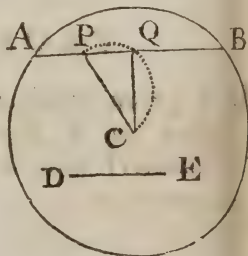
When the point given is placed without the circle the construction will be no ways different.

PROBLEM XXIII.

Through a given point P , so to draw a line AB , that the parts AP , BP , intercepted by that point and the circumference of a given circle shall have a given difference, DE .

CONSTRUCTION.

From the center C , draw CP , upon which let a semi-circle PQC be described, and in it apply PQ , equal to half DE , producing the same, both ways, to meet the circumference in A and B ; so shall $AQ=BQ$ (by 2. 3.); and therefore PB ($=BQ+PQ=AQ+PQ=AP+2PQ$) $=AP+DE$, which was to be done.

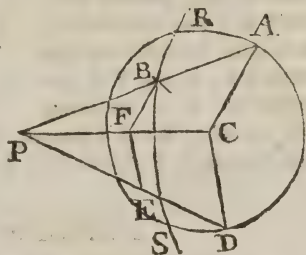


PROBLEM XXIV.

From a given point *P*, to the circumference of a given circle *C*, to draw a right-line *PBA*, so as to be divided in a given ratio by a line *RBS* of any kind given in position.

CONSTRUCTION.

To any point *D* in the circumference, draw *PD*, which divide at *E* in the ratio given; and, having drawn *PF* and the radius *CD*, parallel to the latter, draw *EF*, meeting the former in *F*; from whence, through *R*, apply $FB = FE$; then through *B* draw *PA*, and the thing is done.



DEMONSTRATION.

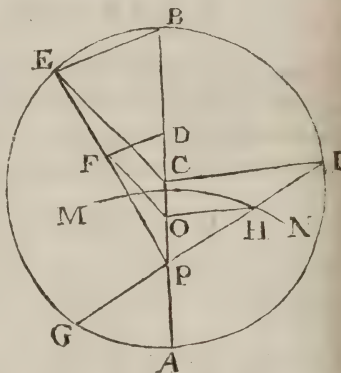
Let *CA* be drawn. Then, because of the parallel lines *CD*, *EF*, it will be, $CD (AC) : EF (CB) :: PC : PF$; and so the triangles *PAC*, *PBF* will likewise be equiangular (*by 16. 4.*) And therefore $PB : BA :: PF : FC :: PE : ED$ (*by 13. 4.*) *E. D.*

PROBLEM XXV.

Through a given point *P*, to draw a line *GH*, terminating in the circumference of a given circle *BGA*, and in a line *MN* of any kind, given by position; so that the rectangle under the two parts *PG*, *PH*, shall be of a given magnitude.

CONSTRUCTION.

Through P draw the diameter AB, in which let PD be so taken (*by* 7. 6.) that $PA \times PD =$ the magnitude given. From any point E in the circumference, draw EP, and the radius EC; and, having joined BE, draw DF parallel to BE, and from its intersection with PE, draw FO parallel to EC, meeting PB in O; from whence, to MN, apply $OH = OF$; and through P draw HG for the line required.



DEMONSTRATION.

Let PH be produced (if necessary) to meet the circumference of the circle in I; and let C, I be joined.

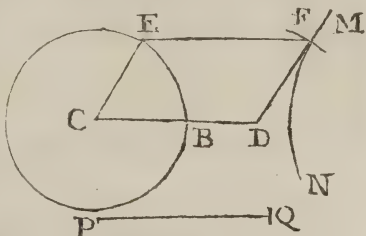
The lines OF, CE being parallel, thence will $PO : PC :: OF (OH) : CE (CI)$; and therefore OH and CI will likewise be parallels (*by* 16. 4.) Therefore $PH : PI :: PO : PC :: PF : PE :: PD : PB$; whence, alternately, $PH : PD :: PI : PB :: PA : PG$ (*by* 21. 3. and 10. 4.) and consequently $PH \times PG = PD \times PA$. Q. E. D.

PROBLEM XXVI.

From the circumference of a given circle C, to a line MN of any kind, given by position, so to draw a right-line EF, as to be both equal and parallel to a given right-line PQ.

CONSTRUCTION.

From the center C, draw CD equal and parallel to PQ; and, from D to MN, apply DF = the radius CB; draw CE equal and parallel to DF; then, EF being drawn, it will (by 26. 1.) be equal and parallel to CD; which was to be done.

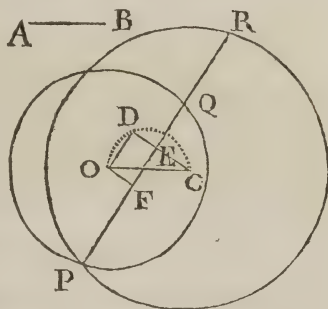


PROBLEM XXVII.

From the point of intersection P of two given circles O and C, so to draw a line PR, that the part QR intercepted by the two peripheries, shall be equal to a given line AB.*

CONSTRUCTION.

Upon the line OC joining the two centers, let a semi-circle ODC be described, in which apply OD = $\frac{1}{2}$ AB; and parallel hereto draw PR, and the thing is done.



DEMONSTRATION.

Let CD and OF parallel to CD, be drawn, meeting PR in E and F. Then, the angle ODC being a

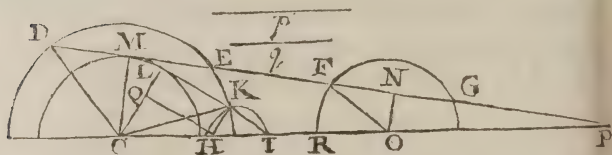
* A problem of Apollonius, *de Inclinacionibus*; See Horsley's and Burrow's *Restitutions* of that Treatise, 4to, 1770, and 1779.

right one (by 13. 3.) and PR parallel to DO , E , and F will also be right-angles, and $EF = DO$ (by 24. 1.). And so, $PE - PF$ being $(= DO) = \frac{1}{2}AB$, it is manifest that $2PE (PR) - 2PF (PQ) = AB$, or that $QR = AB$. *Q. E. D.*

PROBLEM XXVIII.

*From a given point P , in the line passing through the centers of two given circles C and O ; so to draw a line PD , that the parts DE , FG , intercepted by those circles, shall be in a given ratio, (viz. as p is to q .)**

CONSTRUCTION.



Take CH to OR , as p to q ; and CI to OR , as PO to PO : Upon HI let a semi-circle be described, intersecting the circle CDE in K ; through which point draw IKL , and make CL perpendicular thereto; at which distance, from the center C , let a circle ML be described: Then, if from P a line PD be drawn to touch this circle, *the thing is done.*

DEMONSTRATION.

Let CD , CK , HK , FO be drawn, and also CM and ON , perpendicular to PD , and HQ to CL .

The right-angled triangles CDM , CKL having $CD = CK$, and $CM = CL$, have also $DM = LK = Q$ (since, by 13. 3. the angle HKI , as well as L , is right-angle). Moreover (by Construction) $CI : O$

* See Burrow's *Apollonius*, Prob. 3.

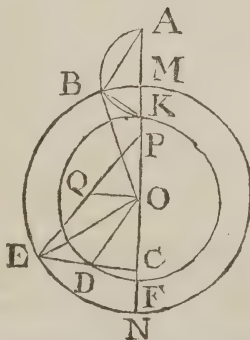
$(OF) :: PC : PO :: CM (CL) : ON$; and so the triangles CIL , OFN (having their sides proportional) must be similar; and consequently OFN also similar to CHQ : whence, as QH (or DM): $FN :: CH : FO$ (OR) $:: p : q$ (by Construction.). *Q. E. D.*

PROBLEM XXIX.

To draw a line EC, to make given angles with a line MN passing through the common center O, of two given circles MEN, KDE, so that the parts CD, ED, intercepted by that line and the two peripheries, shall obtain a given ratio.

CONSTRUCTION.

Let QOM be the given angle, to which ECM shall be equal: In OM produced, let KA be so taken, as to be to the radius OK in the given ratio of ED to CD ; upon which let a segment of a circle ABK be described to contain an angle equal to QOM ; and from its intersection with the circle MEN , draw EA ; parallel to which, draw the radius OD ; and then, through O , draw EC parallel to QO .



DEMONSTRATION.

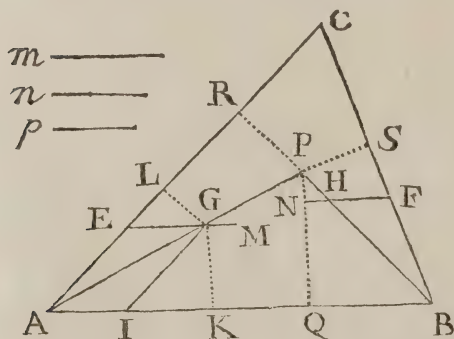
Let BK , BO , EO be drawn, and also EP parallel to DO , meeting OQ and OM in Q and P . Then, the triangles AKB , POQ , having $ABK = POQ$ (by Construction), and $KAB = OPQ$ (by 7. 1.) have their external angles OKB , EQO also equal (by 9. 1.): Therefore, because $EQ (=OD) = OK$, and $EO = BO$, thence is $OQ = KB$ (by 17. 1.); and therefore, as the triangles POQ , ABK are equi-

angular, PQ is also $= AK$: But (because of the parallel lines) $CD : ED :: DO (OK) : PQ (AK)$.
Q. E. D.

PROBLEM XXX.

*To determine a point P, so that three perpendiculars drawn from thence, to as many right-lines AB, AC, BC, given by position, shall obtain the ratio of three given lines m, n, and p respectively.**

CONSTRUCTION.



Take AE and BF each $= m$, and draw EM and FN parallel to AB ; in which take $EG = n$, and $FH = p$. through G and H draw AP and BP , and the point of concourse P , will be that which was to be determined.

DEMONSTRATION.

Upon the sides of the triangle ABC let fall the perpendiculars GL , GK , PR , PQ , PS ; and let GI , parallel to AC , be drawn.

* See Sir Isaac Newton's *Universal Arithmetic*, Prob. 2^d Edit, 1720.

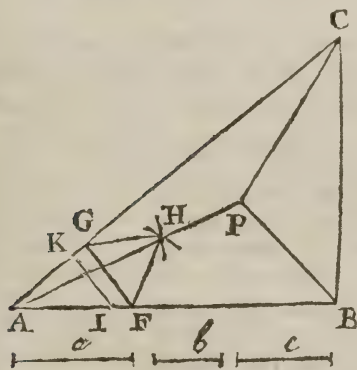
The angle LEG being $= \text{LAK} = \text{GIK}$ (*Cor. 1. to 7.*) and L and K being both right-angles (*by Constr.*), the triangles EGL and GIK are similar; and therefore $\text{IG} (m) : \text{EG} (n) :: \text{GK} : \text{GL} :: \text{PQ} : \text{PR}$ (*by 21. 4.*). And, in the very same manner, $n : p :: \text{PQ} : \text{PS}$. *Q. E. D.*

After the same way, the Problem may be constructed, when the lines drawn from P are required to make any given angles with the lines upon which they fall.

PROBLEM XXXI.

To determine a point P, so that three lines PA, PB, PC, draw from thence to three given points A, B, C, shall obtain the ratio of three given lines a, b , and c , respectively.*

CONSTRUCTION.



Having joined the given points, in AB take $\text{AF} = a$, and $\text{AI} = c$; make the angles AFG and AIK equal,

* See Sir Isaac Newton's *Arithmetic*, Prob. 28.—App. to Simpson's *Algebra*, Prob. 53.—Simpson's *Exercises*, Prob. 56. Vieta, *p. Math.* p. 345.

each, to ACB ; and from the centers F and G , with the radii b and AK , let two arcs be described, intersecting in H ; from which point draw HF and HA ; then draw BP to make the angle $ABP = AHF$, and it will meet AH (produced) in the point P , required.

DEMONSTRATION.

Let BP , CP , and GH be drawn.

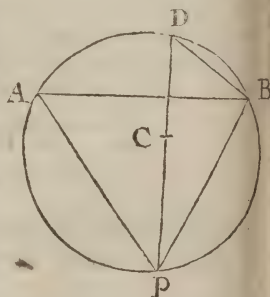
The triangles ABP , AHF being equiangular, it will be, $AP : BP :: AF (a) : FH (b)$; also $AB : AP :: AH : AF$; and $AB : AC :: AG : AF$ (because ABC and AGF are likewise equiangular (*by Constr.*). Now, seeing the extremes of the two last proportions are the same, the four means, AP , AC , AG , AH (*by 10. 4.*) will therefore be proportionals; and so, the triangles ACP , AHG being equiangular (*by 15. 4.*) it will be $AP : CP :: AG : GH (AK) :: FA (a) : AH (c)$. *Q. E. D.*

PROBLEM XXXII.

*To determine the position of a point P , at which, lines drawn from three given points A , B , C , shall make given angles, one with another.**

CONSTRUCTION.

Draw AB , upon which (*by 22. 5.*) let a segment of a circle APB be described, capable of containing the given angle which the lines drawn from A and B are to include; and let the whole circle be completed; make the angle ABD equal to that which the lines drawn from A and C are to include; and



* Simpson's *Algebra*, Appendix. Prob. 48, 49, Vieta, *Opera* page 345.

from the point D, where BD meets the circumference, through C, let DP be drawn, meeting the circumference in P; *which is the point required.*

DEMONSTRATION.

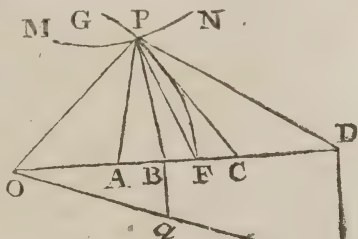
For, AP and BP being drawn, the angle APC will be = the given angle ABD (by 11. 3.) both standing on the same arch AD: And APB is also of the given magnitude by construction.

PROBLEM XXXIII.

Any two unequal segments AB, CD of a right-line AD being given, as well in position as length; to determine a point P in a line MN of any kind given by position; at which the two angles APB, CPD, subtended by those segments, shall (if possible) be equal to each other.

CONSTRUCTION.

Make Ba and Dc parallel to each other, taking the former = BA, and the latter = DC; and through their extremes draw caO, meeting DA produced, in O: Take OF a mean-proportional between OB and OC; and from the center O, with the interval OF, let a circle FG be described; which (when the Problem is possible) will cut (or touch) MN, and the point of intersection P, will be *that required.*



DEMONSTRATION.

Let PO, PA, PB, PF, PC, and PD be drawn.

Because OD : Dc (DC) :: OB : Ba (BA) it will be (by division) OD : OC :: OB : OA; and consequently $OD \times OA = OC \times OB = OP^2$ (by Constr.)

DEMONSTRATION.

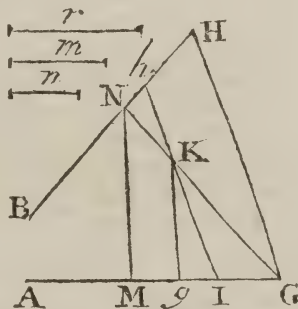
Because of the parallel lines, $AI : GH (: : ID : D) :: BP : BE$; whence, alternately, $AI : BP :: H (p) : BE (q)$. In the very same manner, $AI : N :: p : r$. Q. E. D.

PROBLEM XXXV.

Between two lines AG , BH , given both in position and length, to draw a line MN , which shall be in a given ratio to each of the segments MG , NH , cut off from the two given lines.

CONSTRUCTION.

Let the given ratio of MN , MG and NH be that of r , m and n , respectively: In AG and BH take $Gg=m$, and $Hh=n$; and, having drawn GH , parallel to it draw hI ; to which, from g , apply $gK=r$; draw GKN , meeting BH in N , and parallel to gK draw MN , and the thing is done.



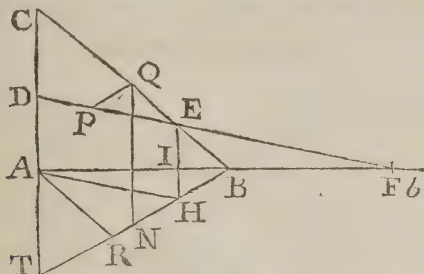
DEMONSTRATION.

Because of the parallel lines, it will therefore be $MN : GM :: gK (r) : Gg (m)$; and likewise $MN : gK (r) :: GN : GK :: NH : Hh (n)$; which last (by alternation) is $MN : NH :: r : n$. Q. E. D.

PROBLEM XXXVII.

Through a given point P , to draw a line $DPEF$, to cut three lines AC , CB , ABb , given by position; so that the parts DE , EF intercepted by those lines, shall obtain a given ratio.*

CONSTRUCTION.



In CA produced, take AT to CA in the given ratio of DE to EF ; and, having joined B, T , draw PQ parallel thereto; and from its intersection with BC , draw QN parallel to CT ; also draw AR parallel to BC : And in RB take RH (by 18. 5.) so that $NH \times RH = PQ \times TR$: Then draw AH , and $DPEF$ parallel to it, and the thing is done.

DEMONSTRATION.

Because (by Constr. and 10. 4.) $NH : TR :: PQ : RH :: QE : AR$ (by 14. 4.) it follows (by alternation) that $NH : QE :: TR : AR :: TB : BC$ (by 12. 4.) $BN : BQ$: Therefore EH (when drawn) will be parallel to QN or AD (by Cor. to 12. 4.); and so, $DEHA$ being a parallelogram, we have again (by similar triangles) as $DE (AH) : EF :: IH : IE :: AT : AC$. Q. E. D.

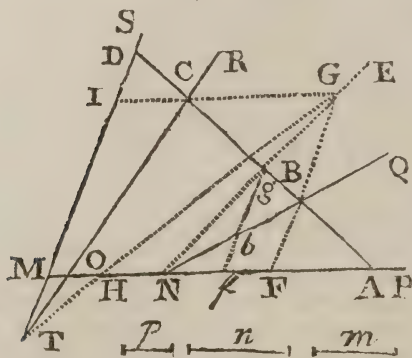
When the segments AD , BE , cut off from the lines AC , BC , are required to be in a given ratio (instead of DE and EF), the construction will be the same; provided that CT be taken to BC in the ratio given. For, then $AD (EH) : BE :: CT : BC$.

* See David Gregory's *Astronomy*, B. V Prob. 8.

PROBLEM XXXVIII.

*To draw a line ABCD to cut four other lines MONP, NQ, ROT, SMT, given by position; so that the three parts AB, BC, CD, intercepted by those lines, may obtain the ratio of three given lines m, n, and p, respectively.**

CONSTRUCTION.



From any point f in NP draw fbg parallel to TMS intersecting NQ in b ; and, having taken bg to bf , in the given ratio of n to m , draw NgE: Moreover, having taken OH to MO in the given ratio of n to p , draw THG; and from (G) its intersection with NE, draw GI and GF parallel to MP and ST, respectively, cutting NQ and TR in B and C; through which two points draw ABCD, *and the thing is done.*

DEMONSTRATION.

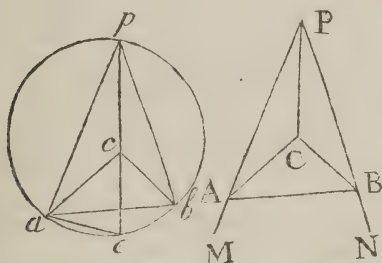
By similar triangles $\left\{ \begin{array}{l} AB : BC :: BF : BG :: bf : bg :: m : n; \\ BC : CD :: CG : CI :: OH : MO :: n : p. \\ Q. E. D. \end{array} \right.$

* See Sir I. Newton's *Principia*, Cor. to Lem. 27, B. I, and Prop. 41, B. III.—David Gregory's *Astronomy*, B. V. Prop. 11.—Playfair on *Porisms*; and Leslie's *Geom. Analysis*, Prop. 25.

PROBLEM XXXIX.

To draw two lines CA , CB , from a given point C , to terminate in two other lines PM , PN , given by position, and which, together with the line AB , joining their extremes, shall form a triangle, ABC , similar to a given one, abc .

CONSTRUCTION.



Upon ab let a segment of a circle apb be described, so contain an angle equal to MPN , and let the whole circle be completed; draw PC , and also ae , making the angle $bae = CPN$, and intersecting the periphery in e ; and through c draw ep , meeting the periphery in p ; make the angles PCA and PCB respectively equal to pca , and pcb ; then join A , B , and the thing is done.

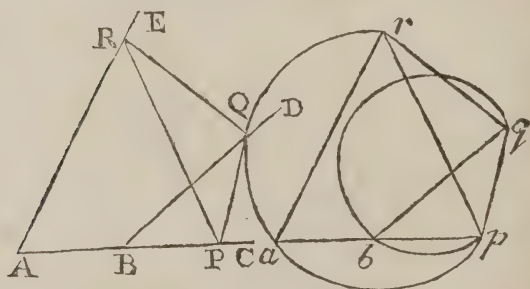
DEMONSTRATION.

If pa and pb be drawn; then will the angle bae (CPB) $= cpb$, both standing on the same arch be ; therefore, APB being also $= apb$ (by Constr.), the remaining angles APC and apc must consequently be equal; whence, as $PCA = pca$, and $PCB = pcb$ (by Constr.), the triangles APC , apc , and BPC , bpc are equiangular; and therefore $AC : ac$ ($: PC : pc$) $: : CB : cb$. And so the triangles ABC , abc , having the sides about the equal angles ACB , acb , proportional, are similar to each other (by 15. 4.) *Q.E.D.*

PROBLEM XL.

To describe a triangle PQR , equal and similar to a given triangle pqr , which shall have its angular points placed in three right lines ABC , BD , AE , given by position.*

CONSTRUCTION.



Upon the two sides pq , pr , let two segments of circles pbq , par be described, to contain angles respectively equal to CBD and CAE : Then draw pa (by *Prob. 27.*) so that the part ba , intercepted by the two peripheries, shall be equal to BA ; and, having joined b , q , and a , r , make $BP = bp$, $BQ = bq$, $AR = ar$, and let PQ , PR , and QR be drawn for the sides of the triangle required.

DEMONSTRATION.

The triangles PAR , par ; PBQ , pbq , being equal and alike in all respects (by *Constr.*), not only the sides PR , pr ; PQ , pq , but the angles QPR , qpr , will be equal; and, consequently, the two triangles PQR , pqr also equal and like to each other. *Q. E. D.*

* *Principia*, Lemma 26, Book I.

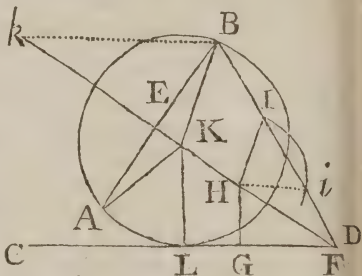
ebg ; ECH , ech , are respectively, equiangular (*by Constr.*) it will be $EG : eg :: EB : eb :: AE : ae$ (*by Constr. and Composition*) $:: EF : ef$ (*by 14. 4.*); and so the triangles GEF , gef (having the sides about the equal angles proportional) are similar. And, in the very same manner, the two remaining triangles EGH , egh (and consequently the whole trapeziums $EFGH$, $efgh$) will appear to be similar. *Q. E. D.*

PROBLEM XLII.

*To describe the circumference of a circle through two given points A, B, which shall touch a right-line CD, given by position.**

CONSTRUCTION.

Draw AB, which bisect in E, by the perpendicular EF, meeting CD in F; from any point H in EF, draw HG perpendicular to CD; and, having drawn BF, to the same apply HI = HG, and parallel there- to draw BK, meeting EF in K; then from the center K, with the radius BK, let a circle be described, and the thing is done.



DEMONSTRATION.

Join K, A, and draw KL perpendicular to CD. Then, because of the parallel lines, $HG : HI :: KL : KB$ (*by 21. 4.*); whence, as HG and HI are equal,

* This, and the seven Problems which immediately follow it, belong to the celebrated work of Apollonius, "*de Tactionibus*," now lost; other constructions may be seen in Vieta's *Apollonius Gallus*, Prob. 2.; and in Lawson's *Tangencies*, Prob. 7.—See also Newton's *Arith. Univ.* Prob. 38.—Simpson's *Algebra*, App. Prob. 43, and Leslie's *Geometry*.

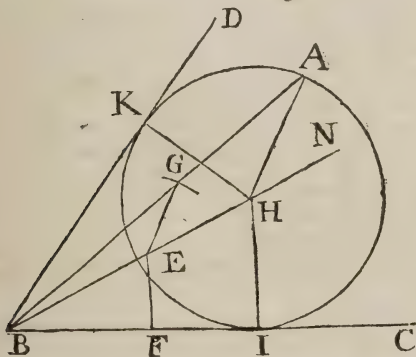
KL and KB are equal likewise: But it is evident, from the construction, that KA is $= KB$; therefore $KB = KL = KA$. *Q. E. D.*

Limitation. Because two equal lines HI , Hi may be applied from H to BF (except when DIH is a right-angle) the problem will therefore admit of two solutions. But when a perpendicular let fall from H upon BF is greater than HG , the problem will be impossible. And the like is to be understood in the construction of the subsequent problems.

PROBLEM XLIII.

To describe the circumference of a circle through a given point A , which shall touch two right-lines BC , BD , given by position.*

CONSTRUCTION.



From the point of concurrence B of the two given lines, draw BA ; and also BN , to bisect the angle BD ; from any point E in BN , upon BC , let fall the perpendicular EF , and to BA apply $EG = EF$, parallel to which draw AH , meeting BN in H ; then

* Vieta, *Apoll. Gall.* Prob. 4.—Lawson's *Tangencies*, Prob. 8.—Newton, *Arith. Univ.* Prob. 44.—Simpson's *Alg.* App. Prob. 42.—Leslie's *Geom.* page 338.

THE CONSTRUCTION OF

from the center H, with the interval HA, let a circle be described, *and the thing is done.*

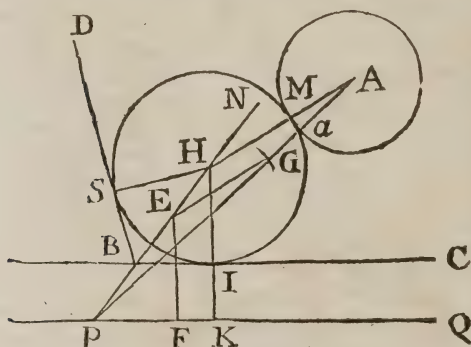
DEMONSTRATION.

Upon BC and BD let fall the perpendiculars HI and HK; which are manifestly equal, because (by *Constr.*) the angle $HBI = HBK$: Moreover, as EF and EG are equal, HI and HA are also equal (by 21. 4.) *Q. E. D.*

PROBLEM XLIV.

*To describe a circle, which shall touch a given circle AAM, and two right-lines BC, BD, given by position.**

CONSTRUCTION.



Draw PQ parallel to BC, at the distance of the radius Aa ; and through the point of concurrence B of the two given lines, draw NBP, bisecting the angle CBD, and meeting PQ in P; moreover, from any point E in PN, upon PQ, let fall the perpendicular

* Vieta's *Apollonius Gallus*, Prob. 5.—Lawson's *Tangencies*, Prob. 9.—Sir Isaac Newton's *Algebra*, Prob. 45.—Leslie's *Geometry*, p. 344.

EF, and from the same point, to PA, apply $EG = EF$; draw AH parallel to EG, meeting the periphery of the given circle in M, and the right-line PN in H, from which last point, as a center, through M, describe the circumference of a circle; and the thing is done.

DEMONSTRATION.

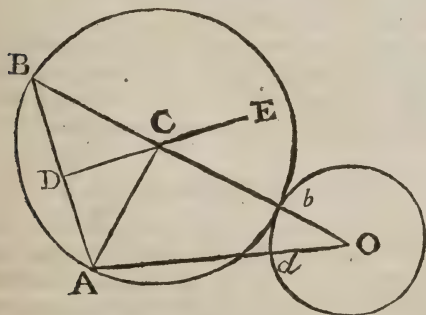
Draw HS perpendicular to BD, and HK to PQ, intersecting BC in I.

Then, because EG and EF are equal (*by Constr.*) IA and HK (*by 21. 4.*) are likewise equal; from which take away $KI = AM$ (*Aa*) and the remainders II, HM will be equal: But, it is evident, that HI is $= HS$, because BH bisects the angle IBS; therefore $II = HM = HS$. Q. E. D.

PROBLEM XLV.

To describe the circumference of a circle through two given points A, B, and which shall also touch another circle Odb, given in position and magnitude.*

CONSTRUCTION.



Bisect the given distance AB with the perpendicular E, in which (*by Prob. 15.*) let the point C be so

* Vieta, Prob. 8.—Lawson, Prob. 12.—D'Omerique, I. 47.—*Mathematician*, Prob. 27.—Newton's *Algebra*, Prob. 46.—*Euclid's Geom.* page 341.—*Math. Comp.* 1811.—Leybourn's *Diaries*, vol. iv. page 265.—Camerer's *Apollonius*, where the Problem is divided into eight cases.

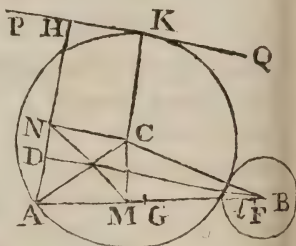
determined, that CO (when drawn) shall exceed CA by the radius Od of the given circle. Then that point, it is manifest, will be the center of the circle to be described.

PROBLEM XLVI.

Through a given point A , to describe the circumference of a circle, which shall touch a given circle B , and also a right-line PQ , given by position.*

CONSTRUCTION.

Make AH perpendicular to PQ , and BD to AH ; and having drawn AB , in it take BF a third proportional to AB and the radius Bd , and let AF be bisected in G : then draw MN (*by Prob. 35.*) so that MN , NH , and MG may be in the same given ratio, among themselves, as BD , BA , and Bd : and at M and N let two perpendiculars be erected on AB and AG ; which will intersect each other in the center C of the required circle.



DEMONSTRATION.

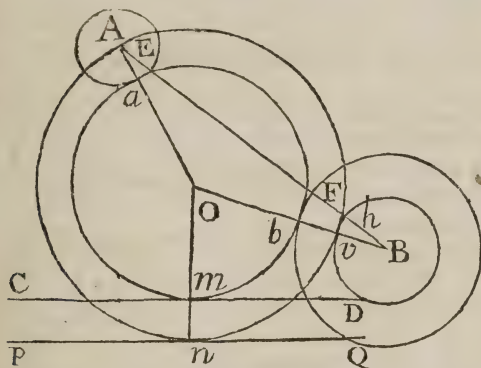
Let CA and CB be drawn, and also CK perpendicular to PQ. Because $MN : NH :: BD : BA$ (*by Constr.*) $:: MN : AC$ (*by 22. 4.*), thence, is NH (CK) $= AC$. And since (*by Constr.*) NH (AC) $: MG :: BA : Bd$, it is also manifest, from the Lemma on p. 215. that $BC = AC + Bd$. Q. E. D.

* Vieta, Prob. 6.—Lawson, Prob. 10.—*The Mathematician*. Prob. 26.—Leslie, page 342;—and Camerer's *Apollonius*, where he distinguishes 15 cases of this Problem.

PROBLEM XLVII.

To describe a circle, *oamb*, which shall touch two given circles *AEA*, *BFb*, and also a right-line *CD*, given by position.*

CONSTRUCTION.



From the radius *BF* of the greater circle, take away *Fh* equal to the radius *AE* of the less; and from the center *B*, with the interval *Bh*, describe the circle *Bhv*; also draw *PQ* parallel to *CD*, at the distance of *Fh* or *AE*: Then, by the last Problem, let the center *O* of a circle be found, whose circumference shall pass through *A*, and touch *PQ*, and *Bhv*; and the same point *O* will, likewise, be the center of the required circle *Oamb*.

DEMONSTRATION.

Draw *On* perpendicular to *PQ*, cutting *CD* in *m*; also let *OA* and *OB* be drawn intersecting the circles *A* and *B* in *a* and *v*: Then, since (by Constr.) $AO = vO = nO$, and Aa (*AE*) = bv (*Fh*) = nm , let these last be respectively taken from the former, and there will remain $Oa = Ob = Om$. *Q. E. D.*

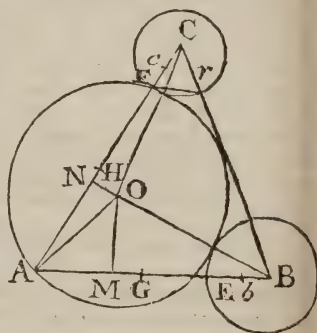
* Vieta, Prob. 7.—Lawson, Prob. 11.—Newton, *Arith. Univ.* Prob. 40.

PROBLEM XLVIII.

To describe the circumference of a circle through a given point A, which shall touch two other circles B and C, given in position and magnitude.*

CONSTRUCTION.

To the centers of the given circles, draw AB and AC; in which take Bb=a third-proportional to AB, and the radius BE; and Cc=a third-proportional to AC and the radius CF: bisect Ab and Ac in G and H, and let Fr be drawn parallel to AB, meeting BC in r. Then (*by Prob. 34.*) draw AO, so that OM and ON being drawn perpendicular to AB and AC, the three lines AO, GM and HN shall have the same given ratio among themselves, as AB, BE and Fr. Then shall the point O be the center of the required circle.



DEMONSTRATION.

For, since (*by Constr.*) $AO : GM :: AB : BE$; and $AO : HN (:: AB : Fr) :: AC : CF$; it is manifest from the Lemma on p. 215, that $OB = AO + BE$; and $OC = AO + CF$. Q. E. D.

* Vieta, Prob. 9.—Lawson, Prob. 13.—Newton, *Arith. Univ.* Prob. 41.—Rob. Simson, *Opera Reliqua, Appendix.*—*Math. Comp.* 1811.—*Correspondance sur l'Ecole Polytechnique*, Tom. 1. et 2.—Leybourn's *L. Diaries*, IV, App.—Camerer's Lemmas to *Apollonius de Tactionibus*, App. Prob. 5, where he shows that this Problem admits of 40 different cases, or varieties, according to the different positions of the given point, with respect to the two given circles.

SCHOLIUM.

In the very same manner, *a circle may be described to touch three given circles* ;* the Problem amounting to no more than, *To find a point from whence lines, drawn to three given points, shall have given*

* See Vieta, Prob. 10.—Lawson, Prob. 14.—Newton, *Principia*, Lemma 16, Book I.—*Arith. Universalis*, Prob. 47.—L'Hospital's *Conic Sections*, Book X, Ex. 4. Cor. I.

“Vieta, in a dispute which he had with Adrian Romanus, a Geometrician of the Low Countries, proposed to him this Problem. The solution which Romanus gave of it, though obvious, was very indifferent, viz. by determining the center of the circle sought, in the point of intersection of two hyperbolas : for, as the Problem is a plane one, it may be solved by plane geometry ; by this, Vieta solved it, and very elegantly, in his *Apollonius Gallus*, printed at Paris, 1600 : his solution is the same as that given in Newton's *Universal Arithmetic*. Another solution may be seen in Lemma 16, Book I. of the *Principia*. (this question being there necessary for some determinations in Physical Astronomy), where Newton, by a remarkable dexterity, reduces the two solid loci of Romanus to the intersection of two straight lines.—Moreover, Descartes attempted to solve this Problem by the Algebraic Analysis, but without success ; for, of the two solutions which he derived from thence, he himself acknowledges (see his *Lettres*, tom. iii, lett. 80, 81), that one furnished him with so complicated an expression, that he would not undertake to construct it in a month ; while the other, though somewhat less complicated, was not so very simple as to encourage him to set about the construction of it.—Lastly, the Princess Elizabeth of Bohemia, who, it is well known, honoured Descartes with her correspondence, deigned to communicate a solution to this Philosopher ; but, as it is deduced from the algebraic calculus, it labours under the same inconveniences as that of Descartes.”—Montucla, *Hist. Math.* Tom. II.

See also Frisi, *Op. Math.* Tom. I, Prob. 20, page 64, Milan, 1782.—*Nova Acta Acad. Petrop.* Tom. 6, p. 95, et seq. where three solutions of this problem are given, namely, one by Euler, which proceeds wholly on the principles of Trigonometry, and two by Fuss, the first of which is purely algebraical, and the second is an analytical process, conducted on trigonometrical principles.

The following Problem was proposed by Descartes to Fermat : it is given here, because of its similarity to the Apollonian Problem.

Suppose four things to be given in position, consisting of Points, Planes, and Spheres, given in Magnitude, which may be taken of any one of these kinds, exclusively, or of any two of the kinds, indifferently, or lastly, of all the three kinds, promiscuously : it is required to describe a Sphere which shall pass through each of the given Points, and also touch each of the given Planes or Spheres.

differences :* Since a point so found, will always be the center of the required circle, as well when the three given circles are to touch that circle *inwardly*, as when they are all required to touch it *outwardly*.

* Two very elegant solutions of this Problem, one by Algebra, the other a geometrical construction, may be seen in the Author's *Select Exercises for Young Proficients in the Mathematics*, 8vo. which is an excellent practical performance, perhaps, indeed, the very best in the English language, which can be put into the hands of young men who wish to acquire a correct and elegant taste, by the study of a masterly specimen of analytical investigation.

THE END OF THE GEOMETRICAL CONSTRUCTIONS.

NOTES,

GEOMETRICAL AND CRITICAL.

AXIOM 10. Book I. What is here laid down as an axiom, would more properly have been made a proposition, had it admitted of such a demonstration, as is perfectly consistent with geometrical strictness and purity. But the laying of one figure upon another, whatever evidence it may afford, is a mechanical consideration, and depends on no *Postulate*.

Theor. 4 and 5. Book I. There is scarcely any thing more obvious to sense, and at the same time more difficult to demonstrate, than the first and most simple properties of parallel lines. Even when we have (in Theor. 4.) proved the possibility of the existence of such lines, we cannot from thence infer, that their distance from one another is every where the same; without having recourse to an Axiom, which, though very evident to sense, cannot be demonstrated. These difficulties only arise from our not having any properties previously demonstrated, by which the progress of a right-line, produced out, can be traced, with respect to its distance from some other right-line assigned; nothing less being required here, than the proportionality of the sides of equiangular triangles, which is not proved before the middle of the Fourth Book, and which depends upon these very principles.

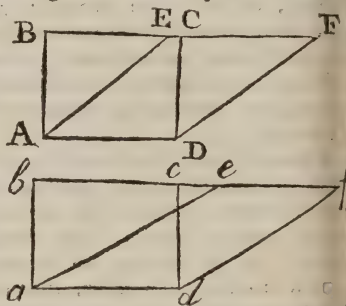
Schol. to Theor. 5. Book I. As there are several conditions requisite to make up the definitions of a rectangle and square, it was necessary to show here, that the several properties ascribed to those figures, are not incompatible one with another. *Euclid* is very strict in this particular, and never undertakes to demonstrate any thing relating to a figure, till he has proved the possibility of the existence of such figure by an actual construction.

Theor. 22. Book I. This proposition, which is not in *Euclid*, is of considerable use, being often wanted in de-

termining the *Maxima* and *Minima*, in Mathematic inquiries.

Theor. 27. Book I. This Theorem, though not in *Euclid* is also very useful, at least, to our design: by it we are not only enabled to divide a right-line into any number of equal parts, without the help of proportions, but also to demonstrate that most important proposition, *That the homologous sides of equiangular triangles are proportional.* It is true the method pursued here, is not exactly conformable to the idea of proportions delivered in the 6th Def. of *Euclid* 5th Book. But, even in that light, the demonstration will be equally easy, without inferring it from the proportionality of triangular spaces.

Theor. 1. Book II. This proposition, which is not mentioned by *Euclid*, may be thought unnecessary; but it must either be demonstrated, or assumed, as it is wanted in almost every proposition of the second book: By means of it, we also arrive at a general, and very easy demonstration of that important Theorem,—*That all parallelograms (AEFD, aefd) which stand upon equal bases*



and have equal altitudes, are themselves equal. For, when it is known that these parallelograms AEFD, aefd, are equal respectively, to rectangles ABCD, abcd, of the same base and altitude (which is proved in Prop. 2.) it is also manifest that they must be equal to one another, as their equal rectangles ABCD, abcd are shown (by *Theor. 1*) to be equal the one to the other. This determination is more general than that given by *Euclid*, in the 36th Prop. of his first book where he demonstrates the equality of parallelograms, whose equal bases are in the same straight line: Which may, perhaps be thought sufficient for the whole; because, if the bases are not in the same right-line, one of the two figures may be conceived to be removed, and so placed, that its base shall be in the same right-line with the base of the other.—But that these were not *Euclid*'s sentiments, is evident from this he hints at no such thing; and had he approved of this sort of demonstration, his 36th Prop. would have been entirely useless; as nothing more (after the 35th) would be necessary, in order to a general demonstration, than barely to place one base upon the other. But it is certain, that this

a kind of demonstration, which *Euclid* never has recourse to, when the thing in hand can be done without it. For which reason I cannot help wondering a little, that that very accurate geometer, *Professor Simson* of *Glasgow*, should make use of it, where (I apprehend) *Euclid* would not. The place I have now particularly in my eye is the addition or it cannot properly be called a corollary) made by him to the first proposition of the sixth book: Which addition would have been quite unnecessary, had what is above remarked, respecting the equality of parallelograms, been fully established in the second book: For then the demonstration, that parallelograms, having equal altitudes, are as their bases, would nothing differ from that by which parallelograms, standing between the same parallels, are proved to be in proportion as their bases.

Theor. 9, 11, 12, 13. Book II. These four Theorems, though not in *Euclid*, are of very considerable use, particularly the two first of them.

Theor 1. Book III. This easy proposition is added in order to give the learner a proper, and more precise idea of the quantity of an angle, and of its division in practical uses.

Theor. 2, 3, 4, 5, 6, 7, 8. Book III. These seven propositions comprehend all that is most material in the first seventeen Theorems of *Euclid's* 3rd book.—As there is nowhere, in this author, so long a run of propositions together that are less entertaining to learners, or of less real importance, than the greater part of the said Theorems; I thought it would be of use to reduce the substance of them into a less compass: And I flatter myself that I have not succeeded ill in this particular.

Theor. 24, 25, 26, 27, 28. Book III. These Theorems, which are all of very considerable use, will not, I flatter myself, appear less plain by being proved independent of proportions, as the demonstrations here given are, not only more concise, but depend also upon fewer principles.

Def. 12 and 13. Book IV. The explication here given is not strictly conformable to the idea of ancient *Commentators*, but is delivered in a sense somewhat more general. With them, *the composition and division of ratios*, extends to those cases only, where the sum, or difference of each antecedent and its consequent, is compared with the consequent. When the antecedent is compared with its excess above the consequent, this they call the *conversion of ratios*. But in such cases where the antecedent is less than the consequent, and where the sum, or difference, of the antecedent and consequent is required to be compared with the ante-

cedent; it does not appear that any *terms* have been given to signify such a comparison. *Professor Simson* thinks that the definitions we have, are not *Euclid's*, but an addition by *Theon*; which to me appears highly probable: That at least seems clear, that these definitions ought either have been extended to a greater number, or else to have been rendered more general. For this reason I have, after the example of modern geometers, given the signification of those terms, so as to include all the several cases. And this, I thought, might be done with the greater propriety, as the truth of whatever is here understood, depends upon the same demonstration.

Theor. 20, 21, 22, 23. Book IV. All these Theorems though not in *Euclid*, are of considerable use: by the former, the demonstration of several others is rendered more easy; and the latter have been applied, to good purpose, in the analytical determination of some difficult geometric problems.

Prob. 3, 4, 5, 6. Book V. In the demonstrations here given, it may be thought, that I have affected an unnecessary exactness, by making them depend on *Axioms* alone. But I was willing that these fundamental propositions should have the same foundation and evidence, as they appear to stand upon in *Euclid*; without referring to any thing derived in virtue of the 4th Postulate. But, whether I judge well or ill in this particular, is of little consequence, as the demonstrations here given are not less plain, and but very little longer, than they otherwise would have been.—The Problems themselves might, indeed, have been given alone with the Theorems, as they became necessary, according to the method pursued by *Euclid*; by which means any objection of this sort might have been avoided. But, besides some small convenience to the learner, there is a real advantage in having the Problems altogether, after the Theorems; since from a great choice of properties, readily demonstrated to our hands, we are often able to arrive at a shorter and better construction than could possibly be given from such properties alone as are antecedent to *Euclid's* solution of the same Problem; his method of writing having obliged him to introduce the leading Problems as soon as possible, in order to evince and establish the consistence of his definitions, and to open his way in a regular manner to the many useful Theorems thereon depending. And it is for this reason alone, that many of his constructions are not so well adapted to practice, as those in common use. Upon which account, some have been precipitate enough to blame him

at seeing, or considering, that such constructions, though not suited to answer every purpose, were the most proper for his plan, and the best that could be given in the places where they stand.

Prob. 8. Book V. The reasoning in this proposition, to prove that the two circles will cut each other, may, some, appear needless. *Professor Simson* has been a little severe upon me, on this head, for attempting to apply, what I thought, a small defect in *Euclid*. "Who so dull (says he), though only beginning to learn the elements, as not to perceive that the circle described from the center F," &c. It is not without a real concern that I here see this able geometer drop his own character so far, as to express himself in a manner so very *ungeometrical*. If the thing is, indeed, easy to be perceived, it must be so, either as an immediate object of the senses, that is, in plain terms, by inspection; or else it must be in consequence of geometrical reasonings antecedent to the thing itself. Now I am clear that he would not be thought to mean the former; and as to the latter, nothing had been given from whence the evidence of the inference could be so clearly seen: For, though it is proved, that any two sides of every triangle are greater than the third side, it would be absurd to urge that conclusion in the case before us; because the question here is, whether a triangle, under certain specified conditions, can or cannot be formed? and, therefore, to conclude any thing from the properties of triangles would be ridiculous, and nothing less than begging the question. That the determination proposed limits the Problem, nobody will dispute: But then, is it not necessary that this should be proved, in *Elements of Geometry*, where the reason for every thing is, or ought to be given? From this consideration, I cannot entirely approve of the emendation proposed by this editor to *Euclid's* 24th Prop. Book V. For, though the addition there made, does indeed restrain the proposition to one case, yet this ought to have been demonstrated, by showing that the point F (vid. p. 29) must, in consequence of such restriction, necessarily fall below the line EG; but this is not done.

Prob. 16, 17, 18. Book V. These three Problems, though not so frequently wanted as some of the preceding ones, are nevertheless of very considerable use. The two last of them are the same, in effect, with the 28th and 29th of *Euclid's* 5th book; but are here put down in a manner rather more commodious.

Axiom, p. 124. This Axiom is substituted instead of the

common definition of equal solids, which, I really think, is too bad to be the work of *Euclid*. "It is not a definition" but a proposition, whose truth or falsehood ought (as "very judicious writer observes) to be demonstrated, not "assumed." Neither is it at all conformable to *Euclid's* manner of writing, where he establishes the foundation on which the equality of plane figures is grounded; which he does, not by means of a definition, but from the application of two of the most simple figures to each other; so that, from the coincidence of their bounds, their equality may appear manifest. And this method we have pursued in treating of solids; without which, a clear and distinct idea of their equality can be no more obtained, than of the equality of plane figures independent of the 4th Prop. of the first book, which is our 10th Axiom. I should have said a good deal more on this head, but I find that *Professor Simson* has already placed this matter in so strong and clear a light, as to render any farther apology, or comment, unnecessary here. Though I must confess, that, had this gentleman's work come into my hands* before the elementary part of my own had been entirely printed off, the definition of similar solids, which I have given, from *Euclid*, would have been delivered under a form somewhat different. For, though it involves no absurdity, as it now stands, yet there are certain cases (but such indeed as do not occur in any Elements of Geometry) where it will not afford the precise idea it ought to convey.

Postulatum, p. 124. The unsatisfactory and inconclusive demonstration given to *Euclid's* 2nd Prop. Book XI. (by *Euclid* himself, or some less skilful editor) seemed to render something of this sort necessary. In that demonstration it is taken for granted, that one part of the triangle, at least, must be in the same plane. But it has been very justly observed, that a curve surface may be bounded by three right-

* This did not happen till the middle of November 1759; when being in town, in company with my bookseller, and being pressed by him to finish, I acquainted him that every thing was actually ready except the Preface, which would cost me some pains, since it would be necessary to obviate some objections, particularly with regard to the reasons for my rejecting *Euclid's* definition of equal solids, and building upon a different foundation. On which, he immediately let me know, that Mr. Rob. Simson had already cleared up that point; and expressed his surprise that a work of so much repute, in which (he told me) my own name was more than once mentioned, had not come into my hands. A copy of which I received from him the next morning. In consequence whereof I changed my first design of writing a long Preface, thinking it would be better to give what I had to offer, in notes, after the example of this Editor.

nes: Nor does it seem easy to form a clear idea, that even part of any one of the three lines will be in the same plane with one of the others, unless by conceiving a plane to be turned about upon the one, till it meets with, or falls upon some point in the other. And I have the satisfaction in this particular, to see my sentiments exactly agree with those of every good judge, whose name I have, more than once, had occasion to mention in these notes. It is true, he makes that a Theorem which I lay down as a Postulate. But, since a plane can no more be turned about upon a line, than a line can be drawn from one point to another, it seemed to me, that the one was as properly a Postulate, as the other. However, whether this be, or be not allowed, is immaterial, as the degree of evidence is precisely the same. There is, indeed, one reason, why this Theorem, or Postulate, ought to have preceded that gentleman's demonstration of Prop. 1. Book XI; It is there wanted: For, in the Corollary, on which that demonstration is made to depend, the lines AB, CD, BC, are supposed to be all in the same plane; which ought by no means to be assumed in the first of the 11th. Euclid's 10th Axiom, which that Corollary is intended to apply, and by which the proposition is usually demonstrated, is not limited by any such restriction.

Theor. 12. Book VII. This proposition is added on account of its use, being the foundation on which the whole *art of perspective* in a manner depends.

Theor. 25. Book VII. From this Theorem, which is very extensive in its application, several others of considerable note may be deduced: one or two of which, for the sake of the learner, I shall here derive, and put down by way of example.

Let A, B, C, D, denote four lines in continual proportion.

Then, $\left\{ \begin{array}{l} A : B :: C : D, \\ \text{since } A : B :: B : C, \\ \quad A : B :: A : B, \end{array} \right\} \text{it follows (from } \textit{Theor. 25.}) \text{ that}$

$$A^3 : B^3 :: CBA : CBD :: A : D \text{ (by 22. 7.)}$$

That, of four lines in continual proportion, the cube of the first is to the cube of the second, as the first line is to the fourth.

Again, let $A : a :: B : b :: C : c$ (where A, a; B, b; C, c, may be supposed to represent the homologous sides of two similar parallelepipeds.)

Then, $\left\{ \begin{array}{l} A : a :: B : b \\ \text{since } A : a :: C : c \\ \quad A : a :: A : a \end{array} \right\} \text{it also follows,}$

that $A^3 : a^3 :: BCA : bca$; or, that similar pa-

rallelepipedons are to one another, as the cubes of their homologous sides.—The proportionality of similar parallelepipedons, described upon proportional lines, is also included in the same Theorem; being no other than that case of it, where the proposed ratios are all equal.

Theor. 5. Book VIII. The demonstration of this Theorem might have been delivered under a form somewhat different, by assuming two other solids (without regard to figure), the one less, and the other greater than the proposed parallelepipedon IP, and proving that the cylinder must, *also*, be greater than the one, and less than the other: which is done by means of *Lem. I.* that is, by taking Pm, or Pt, so small a part of IP, as to be less than the difference between the given parallelepipedon, and either of the said solids: from whence the demonstration will proceed on, in the same manner in which we have given it. But, as these additional considerations would have increased the number of schemes, and lengthened the process, without adding one jot to the degree of evidence, it was thought proper, for the sake of the learner, to omit them.

Theor. 8. Book VIII. This Theorem is not so useful as the Corollaries that follow from it, which are all of very great importance: In the 3rd and 4th of them, the proportion of all kinds of prisms and pyramids is assigned, without the assistance of the usual demonstrations given for this purpose; which, though sufficiently evident in themselves, are often found a little perplexing to learners, on account of the schemes, wherein so great a number of lines is necessary.

Observations on the Doctrine of Proportion.

Nothing, in the course of these notes, has been said relative to the Theorems on Proportions, though so nice and critical a subject, and though the method I have therein pursued may stand in need of some apology. But, indeed, the whole of what I have to offer on this head, was too much to be comprised in the compass of one single note, and could not so properly be delivered in several detached ones: For which reason, I shall here throw together all that I have to advance on that subject.

There are two objections that may be brought against the method in which proportions are treated of in this work; the one, grounded on the impossibility of dividing every magnitude into equal parts; and the other, on the incommensurability of two or more magnitudes of the same kind, when compared with each other. The first of these objections ap-

seems, to me, to have very little weight. For, though a magnitude may be so constituted, that the division of it into an assigned number of equal parts cannot be, *actually*, effected by any geometrical construction; yet it is no less evident, for that reason, that every such magnitude has really its third, fourth, or other assigned part, though we are at a loss how to take it; or, in other words, it seems very clear to conceive, that in every proposed magnitude, whatever its figure may be, a less magnitude is contained, which, repeated an assigned number of times, shall be equal to the magnitude given. If, as the most rigid judges allow, every plane figure is equal to *some* square, and every solid equal to *some* parallelepipedon; then the parts of the square, or parallelepipedon, which are actually determinable by a geometrical construction, will also be like parts of the figure first proposed, and such as we conceive to be taken.

The other objection, depending on the incommensurability of magnitudes, is a matter of real difficulty, which we have taken some pains to obviate in the Scholia to our 3rd and 7th theorems. *Euclid*, himself, seems to have been not a little embarrassed with it, if we may be allowed to judge from the different methods he has left us in his 5th and 7th books; the former whereof, which is suited to include the business of incommensurables, being nothing near so easy and natural as the latter. It has, it is true, the advantage of being general; but, that the principles whereon it is grounded are neither so simple, nor so evident as might be wished for, the many disputes about them, since *Euclid's* time, by geometers of the first rank, will, in a great measure, evince. And farther, it seems sufficiently plain, from *Euclid's* own authority, that he himself was not entirely pleased with his own performance on this head; or that he was convinced, at last, that it had not *every advantage*: For, otherwise, it will be very difficult to account for his having demonstrated many things in his 7th book, by another method, whose demonstrations had been actually given before, in the 5th, under a different form. For these reasons, when I see the extravagant commendations that have been lavished on this book of *Euclid*, I am no farther convinced by them, than that great men may sometimes launch out too far in half of opinions which they have adopted. And I believe that, whoever has read the notes on the 5th book, by that great restorer of *Euclid*, *Professor Simson*, will be apt to conclude, that those high encomiums are a little misapplied. Indeed, if all that is advanced in those notes be allowed of, I think the author of them has proved too much; and this

superb fabric of proportions, reared with so much art, stand upon a tottering foundation. It is not by choice that I go out of my way to play the *critic*; but as the writers against the vulgar and indistinct notion of proportions (as they term it) are very severe in their censures, and assume a great superiority, from the boasted *accuracy of their reasonings*, it may be necessary to show my reader, that, though what he is *here* taught on proportions, is liable to some objections, the method which some so greatly prefer, has *also* its difficulties; and that there are *other objections* to it beside its *obscurity*. And this I shall make appear from the learned commentator's own authority and concessions; and in order thereto, shall first refer to his note on Prop. 10 which proceeds thus: "It was necessary to give another demonstration of this proposition, because that which is in the *Greek* and *Latin*, or in other editions, is not legitimate. For the words *greater, the same, or equal, less* have a quite different meaning when applied to magnitudes and ratios, as is plain from the 5th and 7th definitions of Book 5; by help of these, let us examine the demonstration of the 10th Prop. which proceeds thus, &c. He then goes on in a long note, to show the insufficiency of a demonstration, which had been received, by all, as perfectly genuine and satisfactory; and at last comes to the conclusion: "Wherefore the 10th Proposition is not sufficiently demonstrated. And it seems that he who has given the demonstration of the 10th Proposition, as we now have it, instead of that which *Euclid* or *Eudoxus* had given, has been deceived in applying what is manifest, when understood of magnitudes, unto ratios, *viz.* that a magnitude cannot be both greater and less than another. That the things which are equal to the same are equal to one another, is a most evident Axiom when understood of magnitudes, yet *Euclid* does not make use of it to infer that these ratios which are the same to the same ratio, are the same to one another; but explicitly demonstrates this in Prop. 11 of Book 5. The demonstration we have given of the 10th Prop. is no doubt the same with that of *Eudoxus* or *Euclid*, as it is immediately and directly derived from the definition of a greater ratio, *viz.* 7th of 5."

Here the weight of the objection rests on its not having been proved, that, of three given magnitudes A, B, C, the ratio of A to C could not, at the same time, be both greater and less than that of B to C. But if, in the demonstration here rejected as insufficient, there is any real flaw, it is chargeable on the definition of a greater and less ratio.

the reasoning from it is clear, strong, and perfectly scientific. And I would seriously ask the con-temners of the vulgar and confused notion of proportions, if a definition, by which it cannot be known, whether the ratio of the first to the second of four given magnitudes, may not, at the same time, be both greater and less than that of the third to the fourth, is really calculated to afford those very accurate ideas they pretended to? This commentator has too much penetration not to be aware of the force of this objection, which he has attempted to obviate in one particular case. But the new proposition given by him, for that purpose, ought to have preceded the 10th, and have been demonstrated, independent of it. This he also seems apprized of, when he says, that "it cannot be easily demonstrated without the 10th, as he that tries to do it will find." But, be that as it will, I am not at all clear that this "demonstration of the 10th, is the same with that of *Eudoxus* or *Euclid*." *Euclid*, or (if you please) *Eudoxus*, never (that I know of) refer to any definition, till it has been proved, either by an actual construction, or by some demonstration previous to that in hand, that such definition involves no absurdity or conditions that are incompatible one with another. If, therefore, it was conceived possible, that the definition of a greater and less ratio could involve so great an absurdity, as that, *by it*, the ratio of A to C might at the same time be both greater and less than that of B to C; this point, according to the method prescribed by *Euclid*, ought to have been cleared up, not by means of propositions derived in virtue of that very definition, but by others antecedent thereto, and independent thereupon. And, to me, the 8th Prop. seems the proper place for the doing of this, where it might be easily introduced, either in the Prop. itself, or in the way of Corollary. It is there proved, that if, of three magnitudes A, B, C, the first A is greater than the second B, then certain equimultiples of A and B may be taken such, that being compared with some multiple of C, the multiple of A shall be greater, and that of B less than the said multiple of C. Whence, by the definition of a greater ratio, the ratio of A to C is greater than that of B to C. To which might be added—And because A is greater than B, any multiple whatsoever of A must be greater than the same multiple of B; and, consequently, no equimultiples whatsoever, of A and B, can possibly be so taken, that the multiple of A shall be equal to, or less than some multiple of C, and that of B greater than the multiple of C: for, if the multiple of B be greater than the multiple of C, the multiple

of A, which is greater than that of B, must also be greater than the multiple of C. Wherefore the ratio of A to C cannot (by the definition) be less (as well as greater) than the ratio of B to C.

But, notwithstanding all that has been proved on this head, either here, or by that gentleman himself, the same objection occurs again in Prop. 13, where it remains in its full force. For, though it be allowed, that "there are some equimultiples of C and E, and some of D and F, such, that the multiple of C is greater than the multiple of D, but the multiple of E not greater than the multiple of F;" yet it is not demonstrated, nor in any sort shown, that other equimultiples of those quantities cannot be taken such, that the very contrary shall happen.—If the demonstration of the 10th Prop. has been justly rejected by this gentleman himself, as insufficient, because the impossibility of a contrary conclusion had not been shown; can it be thought that this 13th Proposition is, at this day, sufficiently demonstrated, where the same objection occurs, and that in a much greater latitude? I have a much better opinion of this editor's discernment than to imagine, that his passing this matter over in silence proceeded from his not being aware of the difficulty; but it seems to me, that his great dislike to the *vulgar* idea of proportion (so often testified in the course of his notes) would not permit him to borrow any thing from thence, however evident, and though this objection, that strikes deep at the very root of proportions, might by means thereof be very easily removed. I say, the very root of proportions is deeply struck at in this objection; because both the alternation and equality of ratios (*ex æquali sc. dist.*) are grounded on the said 13th Prop. and which, therefore, till the objection is removed, must be allowed to stand upon an uncertain foundation.

The principle hinted at above, whereby the difficulty might be obviated, is, that if a magnitude of any kind be given, or propounded, there may (or can) be another magnitude of the same kind which shall have to it any ratio assigned. This assumption Dr. *Simson* will by no means admit of (though *Euclid* himself, in Prop. 2. of his 12th book, has used it); and, in a long note on Prop. 18. is angry with *Clavius* for having recourse to it; affirming, "That the demonstration (given by means thereof) is of no force;" and that "the thing itself cannot (as far as he can discern) be demonstrated by the preceding propositions, so far is it from deserving to be reckoned an Axiom, as *Clavius*, after other Commentators, would have it." That the assump-

tion cannot be generally demonstrated by the preceding propositions (nor even by all the propositions in the Elements) I readily assent to; but then, because a thing, exceedingly obvious in its own nature, cannot be demonstrated, is it therefore less proper for an Axiom? I should rather take the other side of the question, and maintain that nothing ought to be made an Axiom, which can be demonstrated. But we are not, it seems, allowed to have any idea of proportion but what is contained in the 6th and 8th (or, as this author makes them, the 5th and 7th) definitions of *Euclid's* 5th book. And, in his note on the new Prop. marked A, he is again displeased with *Clavius*, for thinking it sufficiently evident, *from the nature of proportionals*, that if, of four proportional magnitudes, the first antecedent is greater than its consequent, the second antecedent will also be greater than its consequent. "As if there was (says he) any nature of proportionals antecedent to that which is to be derived and understood from the definition of them." Now I cannot help thinking, with *Clavius*, that there was a nature, or idea of proportion antecedent to that given in the 5th and 8th definitions of *Euclid's* 5th book: For, that mankind, long before the time of *Euclid*, had some way to show or express in what degree one magnitude was greater or less than another, cannot be doubted: And this was the first and natural idea of proportion: And I look upon those definitions, as refinements, only, on the simple and natural idea, in order to take in the business of incommensurables; whereby the original notion is so much obscured, that it requires some skill, even to see that it is at all contained in these definitions. I entirely agree with this gentleman, that every demonstration ought to be strictly derived from principles before established: But then, whether is it more eligible, to have recourse to an Axiom founded (as all other Axioms are) on the evidence of sense and reason, or to an obscure and perplexed definition, which may, for any thing that has been proved to the contrary, involve an absurdity?

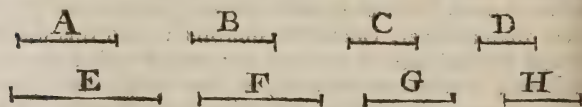
That there is something very ingenious and subtle in the doctrine of proportions, as delivered in *Euclid's* 5th book, cannot be denied. All that I contend for, is, that the principles on which it is built are *obscure*, and not so firmly established, as to authorize its partizans to assume that great superiority they lay claim to, in point of geometrical strictness.

I have intimated above, that the principle is rejected, by which the consistence of the definition of a greater and less ratio might be established, without much difficulty: But I

would not be thought to mean, that the same thing cannot possibly be effected any other way, because I am satisfied that it may be done from the consideration of multiples alone: But a demonstration of this sort is not easy.—Were I to treat of proportions from the plan laid down in the 5th book of *Euclid*, I would entirely reject the 10th and 13th propositions, and every thing else founded on the definition of a greater and less ratio, as being of no other use in the *Elements*, than to open the way to those important Theorems on the alternation and equality of ratios, which may be better demonstrated without them, from the definition of equal ratios alone; which, from the conditions of it, can admit of no absurdity; and whose consistence is evinced in Prop. 15. and still more clearly in the first of the sixth.

In the 14th of the 5th, whereon the alternation of ratios is grounded, it is necessary to demonstrate, “That if, of four proportional magnitudes, of the same kind, A, B, C, D, the first be greater than the third, the second shall be greater than the fourth; and if equal, equal; and if less, less.” Which may be very easily done, independent both of the 10th and 13th, in the manner following:

First, let A be greater than C.



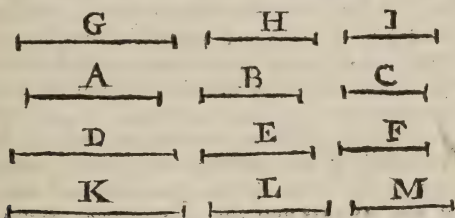
Of A and C (*by Prop. 8.*) let such equimultiples be taken, that the multiple of A shall be greater, and that of C less, than some multiple of B; let E and G be any two such equimultiples of A and C, and F the multiple of B; so that E shall be greater than F, and G less than F; and let H be the same multiple of D, as F is of B. Therefore, because E and G are equimultiples of the first and third, and F and H also equimultiples of the second and fourth; and seeing that (*by Hyp.*) E is greater than F; it is evident, from the definition of equal ratios, that G must likewise be greater than H: Therefore much more shall F (which exceeds G) be greater than H; whence also B shall be greater than D (*by Ax. 4.*) B and D being like parts of F and H.

When A is less than C, it will be demonstrated in the same manner, that B is also less than D. But when A is equal to C, no new demonstration is necessary; since neither the 10th nor the 13th have any thing to do in this case.

Again, in the 20th Prop. (in which the 10th and 13th also enter) we are to prove, “That if there be three magnitudes (A, B, C) and other three (D, E, F) which taken

“two and two have the same ratio ($A : B :: D : E$, $B : C :: E : F$) if the first (A) be greater than the third (C), the fourth (D) shall be greater than the sixth (F); and if equal, equal; and if less, less.” Which may likewise be done without the assistance of either the 10th or the 13th, in the same manner, above specified.

For if A be greater than C , then of A and C (*by Prop. 8.*) such equimultiples may be taken, that the multiple of A shall be greater, and that of C less, than some multiple of B ; let G and I be two such equimultiples of A and C ,



and let H be the multiple of B , so that G shall be greater than H , and I less than H ; moreover take L the same multiple of E , as H is of B ; and K and M the same equimultiples of D and F , as G and I are of A and C . Therefore, since of the four proportionals A, B, D, E , equimultiples G, K of the first and third, and equimultiples H, L of the second and fourth, are here taken, it is manifest, from the definition of equal ratios, seeing G is greater than H (*by Hyp.*) that K must also be greater than L . And in the very same manner, because B, C, E, F are proportionals, and H is greater than I , L will likewise be greater than M . Therefore much more shall K , which exceeds L , be greater than M . And consequently (*by Ax. 4.*) D shall also be greater than F . —When A is less than C , the demonstration is the same: The other case, when A is equal to C , does not require, nor indeed admit, of any improvement.

Note to Problem XXXVIII.

As the history of this problem is not a little curious, the Editor presumes that the following account of it may be acceptable.

“Sir Isaac Newton having demonstrated that the trajectory of a comet is a parabola, reduced the actual determination of the orbit of any particular comet, to the solution of a geometrical problem, depending on the properties of the parabola, but of such considerable difficulty, that it is necessary to take the assistance of a more elementary problem,

in order to find, at least nearly, the distance of the comet from the earth, at the times when it was observed. The expedient for this purpose, suggested by *Newton* himself, was to consider a small part of the comet's path as rectilineal, and described with a uniform motion, so that four observations of the comet being made at moderate intervals of time from one another, four straight lines would be determined, *viz.* the four lines joining the places of the earth and comet, at the times of observation, across which, if a straight line were drawn, so as to be cut by them into three parts, in the same ratios with the intervals of time above mentioned, the line so drawn would nearly represent the comet's path, and by its intersection with the given lines, would determine, at least, nearly the distances of the comet from the earth, at the times of observation.

The geometrical problem here employed, of drawing a line to be divided by four other lines given in position, into parts having given ratios to one another, had been already solved, by *Dr. Wallis* and *Sir Christopher Wren*, and to their solutions *Sir Isaac Newton* added three others of his own, in different parts of his works; yet none of these geometers observed that peculiarity in the problem which rendered it inapplicable to astronomy; this was first done by *M. Boscovich*, but not till after many trials, when, on its application to the motion of comets, it had never led to any satisfactory result: the errors it produced, in some instances, were so considerable, that *Zanotti*, seeking to determine by it, the orbit of the comet of 1739, found that his construction threw the comet on the side of the sun opposite to that on which he had actually observed it. This gave occasion to *Boscovich*, some years afterwards, to examine the different cases of the problem, and to remark, that, by a curious coincidence, this happened in the only case which could be supposed applicable to the astronomical problem above mentioned; in other words, he found, that, in the state of the data which must there always take place, innumerable lines might be drawn, that would be all cut in the same ratio, by the four lines given in position. This he demonstrated in a dissertation published at Rome, in 1749, and since that time in the third volume of his *Opuscula*. A demonstration of it, by the same author, is also inserted at the end of *Castilleoneus's* Commentary of the *Arithmetica Universalis*, where it is deduced from a construction of the general problem, above given in the text, by *Mr. Simpson*.—See the late Professor *Playfair's* ingenious paper "*On the Origin and Investigation of Porisms*," Edinb. Trans. Vol. III.

EDITOR.

APPENDIX.

On the Analytical and Synthetical Modes of Reasoning made use of in the Mathematics.

There are two methods of reasoning made use of in the mathematical sciences—the synthetical and the analytical. By referring back to the Greek origin of the words synthesis and analysis, we shall find that one signifies composition, and the other resolution, or decomposition: nothing appears clearer, at the first glance, than these denominations, and we easily conceive, that the methods they denote, are the reverse, one of the other.

The Elements of Euclid are according to the synthetic method. That author having formed certain axioms, and made certain demands, advances his propositions, which he proves by means of what precedes, and thus continually passes from simple to composite, which is an essential character of synthesis.

The first usage of analysis, in geometrical researches, is attributed to Plato. By this method, we suppose the proposed problem is already solved, from which it results, that one condition is fulfilled, or, what comes to the same, that an equality takes place among certain magnitudes given, and others sought. It is by finding out the consequences of the condition which we supposed fulfilled, or of the equality which is the consequence of it, that, in the end, we discover the unknown quantity, or trace the proceeding which it is necessary to follow, in order to perform what is demanded.

The following are the definitions which Vieta has given of analysis and synthesis, after Theon, a geometrician of Alexandria, who, from his living so much nearer the times of the ancients, was better able to judge of their opinions than we are.

“ Est veritatis inquirendæ via quædam in mathematicis, quam Plato primus invenisse dicitur, a Theone nominata analysis, et ab eodem definita, adsumptio quæsitæ tamquam concessi per consequentia ad verum concessum. Ut contra synthesis, adsumptio concessi per conse-

“quentia ad quæsitum sinem et comprehensionem.”—*Vieta Opera*, p. 1.

There is likewise a definition of analysis and synthesis given in the preface to the 7th Book to Pappus's Mathematical Collections, which is as follows:—

“Analysis is the way, which, proceeding from the thing demanded, arrives, by means of certain established consequences, to somewhat known before, or placed among the number of principles acknowledged as true. This method makes us go from a truth, or proposition, through all its antecedents; and we call it *analysis* or *resolution*, or an *inverted solution*. In synthesis, on the contrary, we begin from the proposition last found in analysis, ordering properly the above antecedents, which now present themselves as consequents, and by combining them among themselves, we arrive at the conclusion sought, from which we proceeded in the first case.

“We distinguish two sorts of analysis: in one, which may be named *contemplative*, it is proposed to discover the truth or falsity of an advanced proposition; the other is related to the solution of problems, or the research of unknown truths.

“In the first, by assuming as true, or as formerly known, the subject of the advanced proposition, we proceed, by the consequences of the hypothesis, to something known; and, if the result is true, the proposition advanced is likewise true. The direct demonstration is lastly formed, by taking, in an inverse order, the different parts of the analysis. If the consequence to which we arrive at last, is found to be false, we conclude that the proposition analysed is likewise false.

“When we have to solve a problem, we suppose it to be already known, and we proceed upon this supposition till we arrive at something that is really known. If the last result which we can obtain, is comprised in the number of what geometricians call *known truths*, the question proposed can be resolved. The demonstration (or more properly the construction) is then formed, by taking, in an inverse order, the different parts of the analysis. The impossibility of the last result of the analysis, will prove evidently, in this case, as in the preceding one, that of the thing demanded.

“There is likewise, in the solution of every problem, what is called the *Diorism* or *Determination*, that is to say, the reasoning by which we show when, in what man-

ner, and how many different ways, the problem can be solved."

The demonstration of Theorems, in the manner called *reductio ad absurdum*, is, properly speaking, an analytical proceeding; for we suppose the converse proposition is true, and, by seeking consequences which are absurd, make it appear that the hypothesis is so likewise.

The characteristics of synthesis and analysis, it is hoped, are rendered tolerably clear by the preceding description. In the *first* method, the proposition enunciated is always the last consequence of the chain of reasonings which forms the demonstration: it is a composition; for we add principle continually to principle, until we come to this consequence.

In *analysis*, on the contrary, by supposing the question resolved, we take in the whole of the subject, and, making it pass with different forms, or making divers traductions of the enunciations, we come to the solutions sought.

But, in the solution of problems, before we begin our analysis, some previous preparation, or mental contrivance is commonly necessary, in order to form a connexion between the *data* and *quæsitæ*, which cannot be brought within any regular rules, being various, according to the nature of the problems proposed. Thus, right-lines must sometimes be drawn in particular directions, or of particular magnitudes, bisecting, perhaps, a given angle, or falling perpendicularly on a given line; or tangents must be drawn from a given point to a given curve; or circles must be described from a given center with a given radius, or touching given lines, or given circles, and other similar operations. And as these antecedent preparations cannot be distinctly pointed out for every particular case, it is here that the learner will have an opportunity of showing his taste and acuteness in choosing the best *effectuations* on which to found his analysis, and thence to obtain the most elegant construction and demonstration the problem admits of; for the solution of the same problem may often be derived in a variety of ways; and however apt some may be to ridicule the notion of *taste*, when applied to subjects of this kind, all persons imbued with a genuine relish for the mathematical sciences, well know, that as much real taste may be shown, in being able to distinguish between the different degrees of elegance in mathematical investigations, as a student in the *Belles Lettres*, or a professed critic can show, by a due relish for the *sublime* or *beautiful* in *poetry* or *oratory*.

A Brief Account of the lost Writings of EUCLID, APOLLONIUS, and others, on the GEOMETRICAL ANALYSIS OF THE ANCIENTS, and of the several attempts of the Moderns to restore them.

PAPPUS* of ALEXANDRIA, in the preface to the seventh book of his very valuable *Mathematical Collections*, has enumerated those works which treated of Analysis, under the name of "*Resolution Books*," or "*Books of Solutions*," which were considered by the ancients as forming a kind of *second elements*, on which they appear to have set a high value.

These tracts are intituled *De Sectione Rationis, de Sectione Spatii, de Sectione Determinatâ, de Tactionibus, de Inclinationibus*, and *De Locis Planis*,—each in two books, and all of them the productions of the elegant and prolific genius of *Apollonius*; to which may be added *Euclid's Book of Data*, and his three books *De Porismatibus*;—none

* PAPPUS was a very eminent mathematician of Alexandria in Egypt, towards the latter part of the fourth century: he is particularly mentioned by Suidas, who says he flourished under the Emperor Theodosius the Great, who reigned from the year 379 to 395 of Christ: his writings show him to have been a consummate mathematician. Many of his works are lost, or at least have not yet been discovered. The principal of these are, his *Mathematical Collections*, in 8 books, the first of which, and part of the second, are now lost. He wrote also *A Commentary on Ptolemy's Almagest*; *An Universal Chorography*; *A Description of the Rivers of Lybia*; *A Treatise of Military Engines*; *Commentaries on Aristarchus of Samos, concerning the Magnitude and Distance of the Sun and Moon, &c.*; of these there have only been published, the *Mathematical Collections*, in a Latin translation, with a large commentary, by Frederic Commandine, in folio, at Pisa, in 1588; and a second edition of the same, also in folio (but less correct than the first) at Bologna in 1660. In 1644, Mersenne exhibited a kind of abridgment of these books, in his *Synopsis Mathematicæ*, in 4to, but this contains only such propositions as could be understood without figures. In 1655 Meibomius gave some of the Lemmata of the 7th book, in his *Dialogue upon Proportions*. In 1688 Dr. Wallis printed the last 12 propositions of the second book, at the end of his *Aristarchus of Samos*. In 1703, Dr. David Gregory gave part of the Preface to the 7th book, in the *Prolegomena* to his *Euclid*. And in 1706 Dr. Halley gave that Preface entire, in the beginning of his edition of *Apollonius on the Section of Ratio*; detached portions of the preface have also been given by Snellius, Dr. Simson, and Camerer.

An analysis of the contents of the *Mathematical Collections*, which are exceedingly curious, may be seen in Dr. Hutton's *Mathematical Dictionary*, art. *Pappus*; and another, much more minute and extensive, in the Rev. Dr. Trail's *Life of Robert Simson*.

of these works, however, have come to our hands, except the treatise *De Sectione Rationis*, and Euclid's *Data*.

These books of Analysis contained complete investigations, both by resolution and composition, in all their possible cases, of a great variety of propositions of a very general nature, and of frequent recurrence in geometrical inquiries. In the solutions of such problems, the geometers of antiquity proceeded with the utmost caution, and were careful to remark every particular case, that is to say, every change in the construction, which any change in the state of the *Data* could produce. The different conditions from which the solutions were derived, were supposed to vary one by one, while the others remained the same; and all their possible combinations being thus enumerated, a separate solution was given wherever any considerable change was observed to have taken place. Nothing, it is evident, that was any way connected with the problem in hand, could escape a geometer who proceeded with such minuteness of investigation.

The use of those general propositions was the more immediate resolution of any geometrical problem, which could be reduced to a particular or known case of any one of them, as by such a reduction, the proposed problem was considered as fully resolved, because it then became only necessary to apply the analysis, composition, and determination, of that particular case of the general proposition, to this particular problem which was shown to be comprehended in it. This apparatus of separate solutions, with the determinations of every possible case, which, to the advantageous use of these general problems, is essentially necessary, may seem forbidding, and if regularly perused without examples of its application, may sometimes appear tedious and uninteresting, and this, perhaps, may have created some prejudice against the study of them. Dr. *Halley*, indeed, seems to think that these books were intended merely for the instruction of beginners in the study of the Geometrical Analysis; but it is justly observed by Dr. *Simson*, that, though they may be very advantageously employed for that purpose, yet, it is manifest from their contents, and also from the manner in which they are referred to, in the small remains which we possess of the ancient Geometry, that the chief object of them was what has now been stated. Indeed in *Pappus* there are some examples of problems being resolved by a reduction of them to cases of one or other of these general problems; for instance, the 87th proposition of his seventh book (a problem useful in the treatise on Inclinations,) is by analysis brought to a case of determinate section.

Another example is prob. 164 of the same book, the last Lemma belonging to the *Porisms*, which is a problem resolved by Pappus, by reducing it to a case of the *section of space*, and the composition of the problem is considered as completed, merely by a reference to that particular case. This tract of Apollonius being lost, and no restoration of it having been made at the time of *Commandine*, the case referred to is resolved by him, in his notes on Pappus, though in a very operose and tedious manner. *Schooten* also resolves this problem (*Exercit. Math.* p. 104, 105), and seems to blame Pappus for not solving it directly, but by a reference to a case of the section of space, from which it appears that *Schooten* was not aware of the real purpose which these ancient books were intended to answer.

I.—On the *Data* of *Euclid*.

Euclid's Data is the first in order of the books written by the ancient Geometers to facilitate and promote the Method of Resolution, or Analysis: it consists of a number of propositions, which are designed to show what things can be known, or determined, from those that, by hypothesis, are already known, and is one of the few which have been preserved in the original language.

“ In general, a thing is said to be *given* or *known*, which is either *actually exhibited*, or can be *found out*; that is which is either *known by hypothesis*, or that *can be demonstrated to be known*; and the propositions in the book of *Euclid's Data* show what things can be found out, or known from those which, by hypothesis, are already known; so that in the analysis or investigation of a problem, from the things that are laid down as *known*, or *given*, by the help of these propositions other things are demonstrated to be given and, from these, other things are again shown to be given and so on, until that which was proposed to be investigated or found out in the problem, is demonstrated to be given and when this is done, the problem is solved, and its composition is made, and derived from the compositions of the *Data* which were made use of in the analysis. And thus the *Data* of *Euclid* are of the most general and necessary use in the solution of problems of every kind, and were constantly used by the ancients in their resolutions of problems though it was not their practice to refer to the particular propositions; and this application of the *Data* is continued by such as follow strictly the ancient method of analysis.”

—*On the two Books of Apollonius, de Sectione Rationis,*
or, On the Section of Ratio.

Apollonius wrote a Treatise on the Section of Ratio, which is still extant, and is comprised in two books; the object of this treatise is, the solution of a single problem, subdivided into a multitude of cases, and marked with various limitations; it may be thus enunciated:

“Through a given point to draw a straight line, so as to cut off segments from two straight lines given in position, which segments shall be adjacent to given points in the said straight lines, and shall also have a given ratio to each other.”

The two books on the Section of Ratio, were long supposed to be lost, and accordingly Willebrordus Snellius, son of Rodolphus Snellius, professor of Mathematics at Leyden, tempted, about the year 1607, at the early age of 17, to store them, in a small work published at Leyden in the following year; and the attempt reflects great credit on its venile author, though it wants that purity, fulness, and elegance, which distinguish the geometrical compositions of the ancients. About a century afterwards, the famous Dr. Halley, with much sagacity, and incredible labour, recovered these two books from an Arabic manuscript, formerly in the possession of the learned Selden, but now in the Bodleian Library; this is the only complete work of Apollonius now extant, and the only work in existence from which a correct view of the mode of solving problems among the ancients is to be obtained: a correct Latin translation of this valuable relic, was published by the Doctor in octavo, at Oxford, in 1706.

I.—*On the two Books of Apollonius, de Sectione Spatii,*
or, On the Section of Space.

The next in order, among the analytical writings of the ancient Geometers, is a Treatise on the *Section of Space*, in two books, and likewise by Apollonius.

The general problem which forms the subject of this treatise, is somewhat similar to the last: it may be thus expressed:

“Through a given point to draw a straight line so as to cut off segments from two straight lines given in position, which segments shall be adjacent to given points in the said straight lines, and shall contain a rectangle equal to a given space.”

IV.—*On the two books of Apollonius, de Sectione Determinata, or On Determinate Section.*

After the treatises on the Section of Ratio, and the Section of Space, comes the treatise on Determinate Section, by the author of the two last, and like them, comprised in two books; the object was the solution of the following general proposition:

“To cut an indefinite straight line in one point, so, that of the segments contained between the point of section sought and certain given points in the said line, either the square on one of them—or, the rectangle contained by one of them and a given line—or, the rectangle contained by two of them—may have a given ratio—either, to the square on one of the remaining lines—or, to the rectangle contained by one of them and a given line—or, to the rectangle contained by two of them.”

This is Pappus's enunciation of the general problem, as corrected by Dr. Simson. Montucla divides it into the three following cases:

1. Two points, A and B being given in position, in a straight line, it is required to find a third point, P, in the same line, such, that the square PA may be to the square of PB; or the rectangle under PA, and a given line, D, to the square of PB; or the square of PB, to the rectangle under PB, and a given line, D, in a given ratio.

2. Three points, A, B, and C, being given, in a straight line, it is required to find a fourth point, P, in the same line, such, that the square of PA may be to the rectangle under PB and PC; or the square of PB to the rectangle under PA and PC; or the rectangle under PA and PB, to the rectangle under PC, and a given line in a given ratio.

3. Four points, A, B, C, and D, being given in a straight line, it is required a fifth point, P, in the same line, such, that the rectangle under PA and PB may be to the rectangle under PC and PD; or, the rectangle under PA and PC, to the rectangle under PB and PD, in a given ratio; where P may have either an internal or an external position, with respect to the given points.

These two books on Determinate Section, long since lost, were restored by the younger Snellius,* before mentioned but in a very imperfect manner, without the necessary distinctions of the various situations of the points (called by Apollonius *Epitigmata*, which Snellius, it would seem, did not understand), and without a complete *exposition* of the determinations, which, as is well known, are necessary to the

perfect solution of any problem.*—These books have again been restored by Dr. Simson, who so passionately admired, and so thoroughly understood, the spirit of the ancient Geometry, in a style the most luminous and complete: the learned Professor has even gone farther, and has added a third and fourth book, entirely new:†—Since Dr. Simson's death, two other restorations of the Determinate Section have been published;—one by *Mr. William Wales*, which is annexed to Lawson's Translation of Snellius;‡ and another by *Giannini*,§ an Italian Geometer, who seems to have trodden in the steps of Apollonius more closely than Mr. Wales has done, inasmuch as he has given two solutions of each of the principal problems, one by straight lines—the other by several circles, which, according to Pappus, was the method pursued by Apollonius: with respect, however, to these Restorations, it may be sufficient to observe, that, independent of the curious information in Dr. Simson's valuable Preface, the great superiority of his work, as a Restoration of Apollonius, must be obvious to every Geometrician.

V.—*On the two Books of Apollonius, de Tactionibus, or on Tangencies.*

After the two books of Apollonius on Determinate Section, Pappus places, as the next in order, two others by the same author, on Tangencies, the object of which was the following very general Problem, branched into a great variety of cases, and marked with various limitations:—

Suppose three things to be given in position, in the same plane, consisting of points, straight lines, and of circles given in magnitude, and which may be taken of any one of these kinds, exclusively;—or, of any two of the kinds, indifferently;—or, lastly, of all the three kinds, promiscuously;—it is required to describe a Circle, which shall pass through each of the given points, (when points are given,) and also touch each of the given lines, (whether they be straight lines, or circles, or both).

Snellius's Restoration was translated into English by Mr. Lawson, and published at London, in 1772, in a thin quarto volume.---This and the other Mathematical Works of Mr. Lawson may be had of the publisher.

See his Opera. Reliqua.

The two books of Apollonius Pergæus, concerning Determinate Section, as they have been restored by Willebrordus Snellius; by John Lawson, B. D. To which are added the same two books, by William Wales, being an entire new work, 4to, London, 1772.

See Opuscula Mathematica, auctore Petro Giannini, Parma, 1773.

It is evident that the Data, in this general enunciation may be so combined, taking them *by threes*, as to give rise to ten distinct propositions,—for,

1. Three points may be given
2. Three straight lines | |
3. Two points and a straight line
4. Two straight lines and a point | |
5. Two points and a circle
6. Two circles and a point o o
7. Two circles and a straight line o o
8. A point, a straight line, and a circle | |
6. Two straight lines and a circle | |
10. Three circles o o o

The solutions of the first and second of these Problems are given in Euclid's Elements, (B. IV. Prop. 4 and 5). The following six constitute the first Book of the Treatise on Tangencies, the two remaining Problems (namely, | | o, and o o o,) occupy the second Book.

We are informed by Pappus, that there was another Treatise, containing a collection of Problems, nearly allied to the Tangencies here described, and to which it was considered introductory. The Propositions it contains may all be included in the following general enunciation, which, it will be perceived, is more circumscribed than the former, with respect to Hypotheses, while, at the same time, it includes an additional condition :—

Suppose two things to be given in position, in the same plane, consisting of points, straight lines, and circles, it is required to describe a circle, given in magnitude, that shall pass through the given point, (or points, if such be given) and also touch the given line (or lines, whether straight lines or circles.)

Here it is evident that six distinct combinations may be formed, by uniting the different Data in pairs, and a circle given in magnitude, may be described to answer all the cases of the general problem, which are the following :—

1. When two points are given
2. — two straight lines |
3. — two circles o
4. — a point and a straight line |
5. — a point and a circle o
6. — a circle and a straight line |

The two Books of Apollonius on Tangencies have shared the same fate as most of his other writings, no vestige of either of them has come down to us, excepting the Lemmata and the preceding general description of them, which are

been preserved by Pappus. Aided, however, by this description, several of the Mathematicians of later times have attempted restitutions, either of the whole, or of particular propositions; for an account of the principal of these attempts, see the notes to Prob. 42. . . . 48 (p. 240, &c.) of this work. A treatise on this subject, by John William Camerer, was published at Gotha and Amsterdam, in 1795, (small 8vo,) containing the Lemmas of Pappus, an edition of Vieta's Treatise before mentioned, with notes and additions, and a curious history of the problem, which is highly interesting, from the accounts it contains of the labours of some foreign mathematicians on this problem, which are little known in this country.

VI.—*On the two lost Books of Apollonius, de Inclinationibus, or on Inclinations.*

The Subjects of these Books was the following general problem:—"Two lines being given in position, it is required to insert a straight line between them, of a given length, that shall also tend, or incline, to a given point."

Some cases of this problem evidently belong to the higher geometry, for the given lines may be either straight lines, or circles, or conic sections, or of the higher orders;—it is manifest, therefore, that certain cases will be plane, others solid, and others, again, linear:—those cases, however, in which the Data are limited to straight lines, or to straight lines and circles, admit of elegant solutions by means of the elements only. The following are among the cases which appear to have been investigated by Apollonius:—

1. A circle being given in position and magnitude, it is required to insert a straight line of a given length, that shall also tend, or incline, to a given point.*
2. A circle being given in position and magnitude, and a straight line being drawn perpendicular to its diameter—insert a straight line of a given length, between the perpendicular and the circumference, that shall also tend, or incline, to the extremity of the diameter.†
3. An angle being given, and a point in the straight line bisecting it being also given, to place a straight line, so that

* Ghetaldus, *Apollonius redivivus*, prob. 1.—D'Omerique, *Anal. Géom.* 37. I.—Leslie's *Geometrical Analysis*, 19. II.

† Ghetaldus, prob. 2.—D'Omerique, 38. I.—Simpson's *Exercises*, Prob. 10.—Reuben Burrow's *Inclinations*, prob. 1.—Leslie's *Geom. Anal.* 23. II.

the part intercepted by the sides of the given angle, may be of a given length, and also tend to (*i. e.* pass through) the given point. This problem is sometimes expressed in the following terms;—a rhombus being given, and one of its sides being produced, to insert in the external angle a straight line of a given length, that shall tend, or incline, to the opposite or internal angle.*

4. Two circles, having their diameters in the same straight line, being given, to insert a straight line of a given length between them, that shall tend, or incline, to a given point in either diameter.†

Pappus informs us, that, “the first Book of the Treatise on Inclinations, contained solutions of the problem concerning a circle, in two cases—of the problem concerning a circle and a straight line perpendicular to its diameter, in four cases—and lastly, of the problem of the rhombus, in two cases. The single problem concerning two circles formed the subject of the second book, and was divided into ten cases, and these again into many dioristic subdivisions.”

The lost Treatise on Inclinations was first restored by Marinus Ghetaldus, of Ragusa, (Anno. 1607,) some of whose defects were supplied by Alex. Anderson, of Aberdeen; other solutions of some of the problems, were given by Des Cartes, Gregory St. Vincent, Renaldine, D’Omerique, Huygens, Thomas Simpson, and Dr. Rob. Simson. In the year 1770, Dr. Horsley, late Bishop of St. Asaph, printed, in Latin, a restoration of the two Books on Inclinations, in a quarto volume, and nine years afterwards the same task was performed, with much greater ability, and with far more simplicity and conciseness, by Mr. Reuben Burrow.

On the Porisms of Euclid.

“After Tangencies, we have, in three Books, Euclid’s *Porisms*, a most curious collection of many things which relate to the analysis of the more difficult and general problems, of which, indeed, Nature affords a great plenty. Now these contain a subtile and natural theory, very necessary and universal, and highly entertaining to those who are able

* Ghetaldus, prob. 3. 4.—D’Omerique, 42. I.—Burrow, prob. 2.—Leslie, 24. 25. II.—Huygens, *Opera Varia*, Tom. 1. prob. 6. 7. page 397, &c.

† Alex. Anderson, *Supplementum Apollonii Redivivi*.—Burrow, prob. 5. 7.—Burrow’s *Diary*, 1779.—Gentleman’s *Diary*, 1785.

to understand and investigate them. They are, however, in species, neither theorems nor problems, but in some sort, of a middle nature, between both; so that their propositions may be formed either as theorems or problems; whence it has arisen, that some Geometers have esteemed them in kind to be theorems, and others problems, having respect only to the form of the proposition. But it is manifest from their definitions, that the ancients better understood the difference between these three; for they said, that a *theorem* was that in which something was proposed to be *demonstrated*—a *problem* that in which something was proposed to be *constructed*—but a *porism* was that in which something was proposed to be *investigated*. Now this definition of a porism is changed by the moderns, who could not investigate all of them, but, using these elements, showed only what it was that was sought, but did not investigate it. And though they were confuted by the definition, and by what has been said, yet they formed their definition from an accident, in this manner: ‘A porism is that which is deficient in hypothesis from a local theorem, (i. e. a porism is a local theorem, deficient, or diminished, in its hypothesis.) Now, of this kind of porism, *geometrical loci* are a species, of which there is great store in the books concerning analysis, and, being collected apart from the porisms, are delivered under their proper titles, because this species is much more diffuse and copious than the rest. For of Loci, some are *plane*, some *solid*, and others *linear*, and besides these, there are *loci ad medietates*, (or which arise from proportionals.) The above is part of Pappus’s Account of the Porisms, what follows has suffered so much from the injuries of time, that all we can immediately learn from it is, that the ancients put a high value on the propositions which they called porisms, and regarded them as a very important part of their analysis. Several defects in Pappus’s text, might, however, have been supplied, if the enumeration, which he next gives, of Euclid’s propositions, had been entire; but on account of the extreme brevity of his enunciations, and their reference to a diagram which is lost, and for the constructing of which no directions are given; they are all, except one, perfectly unintelligible. For these reasons, the fragment in question is so obscure, that even to the learning and penetration of Halley, it seemed impossible that it could ever be explained; and he therefore concluded, after giving the Greek text with all possible correctness, and adding the Latin translation, ‘Thus much for the description of Porisms, which can neither be useful to me nor to the reader,

on account of the want of the scheme of which mention is made, from whence many right lines here treated of, without any alphabetical marks, or any other characteristic distinction, are confounded one with another, and also on account of some things, omitted and transposed, or otherwise vitiated, in the exposition of the general proposition, from whence I am not able to guess what Pappus meant. Add moreover, a mode of expression too concise, and which ought never to be used on a difficult subject, such as this is."

It is true, however, that, before this time, Fermat had attempted to explain the nature of Porisms,* and not altogether without success. Guiding his conjectures by the definition which Pappus censures as imperfect, because it defined porisms only "*ab accidente*," viz. "*Porisma est quod deficit hypothesi a Theoremate Locali*," he formed to himself a tolerably just notion of these propositions, and illustrated his general description by propositions that are, in effect, porisms. But the honour of completing the discovery was reserved for Dr. Simson, whose restoration of the Porisms appeared in the Collection of his Posthumous Works, printed in 1776, at the expense of Earl Stanhope. The account Dr. Simson gives of his progress in this work and the obstacles he encountered, will always be interesting to Mathematicians. In the Preface to his Treatise de Porismatibus, he says, "but after I had read in Pappus, that the Porisms of Euclid were a most curious collection of many things that related to the analysis of the more difficult and general problems, I was earnestly desirous of knowing some thing about them, wherefore often, and by various ways I endeavoured to understand and to restore, as well Pappus' General Proposition, lame and imperfect as it was, as also the first Porism of the first Book, which is the only one out of the three books that remains entire: but my labour was in vain, for I made no proficiency; and when these thoughts had consumed much of my time, and at length had become very troublesome, I firmly resolved not to make any inquiry for the future, especially as that best of Geometers, Halley, had given up all thoughts of understanding them; therefore, as often as they occurred to my mind I endeavoured to put them by: yet, afterwards, it happened that they seized me unawares, forgetful of my resolution and detained me so long, until some light broke in, which

* Porismatum Euclidæorum renovata doctrina, et sub forma Isagogæ exhibita. Opera varia, p. 116.

give me hopes of at least finding out Pappus's General Proposition, which, indeed, not without much investigation, I at length restored. Now this, soon after, together with the first Porism of Book I. was printed in the Philosophical Transactions for 1723, No. 177."*

Dr. Simson's restoration has all the appearance of being just; it precisely corresponds with Pappus's description of them. All the Lemmas which Pappus has given for the better understanding of Euclid's Propositions are equally applicable to those of Dr. Simson, which are found to differ from Euclid's theorems precisely as Pappus affirms those of Euclid to have done. They require a particular mode of analysis, and are of immense service in geometrical inquiries.

Dr. Simson defines a porism to be "a proposition in which it is proposed to demonstrate that one or more things are given, between which, and every one, of innumerable other things not given, but assumed according to a given law, a certain relation, described in the proposition, is shown to take place." This definition, it must be confessed, is not a little obscure, but will be plainer if expressed thus:—A porism is a proposition affirming the possibility of finding such conditions as will render a certain problem indeterminate, or capable of innumerable solutions." This definition agrees with Pappus's idea of these propositions, so far, at least, as they can be understood from the imperfect state of the text; for the propositions here defined, like those which he describes, are, strictly speaking, neither theorems nor problems, but of an intermediate nature between both; for they neither simply enunciate a truth to be demonstrated, nor propose a question to be resolved, but are affirmations of a truth in which the determination of an unknown quantity is involved. In as far, therefore, as they assert that a certain problem may become indeterminate, they are of the nature of theorems; and in as far as they seek to discover the conditions by which that is brought about, they are of the nature of problems.

Another celebrated Scotch Professor, the late Mr. Playfair, whose learning embraced a wider range, has, in the third volume of the Edinburgh Transactions, given a comprehensive and luminous view of this intricate subject. There is also an ingenious Paper on Porisms, "with examples of their use in the solution of problems," in the fourth volume of the same publication, by Mr. William Wallace, now Pro-

* Dr. Simson's Restitution of the Porisms was partly translated by Mr. Lawson, and may be had of the Publisher of this Work.

fessor of Mathematics in the University of Edinburgh; and in the Philosophical Transactions for 1798, there is "a Collection of general Theorems, chiefly Porisms, in the high Geometry," by that distinguished ornament of the senate and the bar, Henry Brougham, Esq. M. P. and F. R. S.

On the two Books of Apollonius, on Plane Loci.

One of the most interesting of all the works of Apollonius was that on *Plane Loci*, comprised in two books. The object of this work was "to investigate the conditions under which a point, varying in its position, is yet limited to the description of a straight line, or a circle, (i. e. a plane locus given in position." These books were about the year 1650 restored, in a sort of algebraic form, by Francis van Schooten,* then Professor of Mathematics at Leyden; and more elegantly, though only partially, by the ingenious Fermat, Counsellor of the Parliament of Thoulouse. But these attempts, however skilful, are entirely eclipsed by the finished production *De Locis Planis*,† which Dr. Robert Simson, the great restorer of the Ancient Geometry, published at Glasgow in the year 1749. If this work differ at all from that which it is intended to replace, it seems to do so only by its greater excellence; thus much at least is certain, that the method of the ancient Geometers does not appear to the greater advantage in the most entire of their writings, than in the restoration above-mentioned; and that Dr. Simson has often sacrificed the elegance to which his own Analysis would have led, in order to tread more exactly in what the Lemmata of Pappus pointed out to him as the track which Apollonius had pursued.

The following propositions are those preserved by Pappus as having formed the subject of the Treatise on Plane Loci; many of them, it will be seen, are very general, and branch into a variety of cases, each of which, as was the custom of the ancient Geometers, was separately considered by Apollonius; the investigations may be seen in the restoration of Dr. Simson, already mentioned.

* Schooten, *Exercit. Math.* lib. iii. 4to. Lug. Bat. 1657.

† Fermat, *Opera Varia Mathematica*, fol. Tolos, 1679.

§ Apollonii Pergæi *Locorum Planorum*, lib. ii. restituti a Robert Simson, M. D. Matheseos in Academia Glasguensi Professore, 4to. German translation of this excellent work, by J. W. Camerer, was published at Leipsic, in 1796, in one vol. 8vo. It has also been translated into French, by Simon L'Huilier, of Geneva.

Book I.

1. If two straight lines be drawn, either from the same given point—or, from two given points—or, which are either *directum*—or, parallel—or, comprehend a given angle,—and, if the lines so drawn have—either—a given ratio to one another—or, contain a rectangle equal to a given space—then, if the extremity of one of them terminate in a plane locus given in position—the extremity of the other will also terminate in a plane locus given in position—sometimes of the same species as the first—sometimes of a different species—sometimes similarly situated with respect to a given line—sometimes situated in a contrary manner.—(Simson, prop. 4.....19. pa. 3.....32.)
2. If one extremity of a straight line given in magnitude, be drawn parallel to a straight line given in position—terminate in a straight line (or a circle) given in position, then the other extremity terminate in a straight line (or a circle) given in position.—(Simson, prop. 20, 21. pa. 33...35.)
3. If two straight lines—either having a given ratio to each other—or, of which one is given, together with that to which the other has a given ratio—be drawn from a certain point, at given angles with two straight lines—either parallel—or intersecting each other—that point will be situated in a straight line given by position.—(Simson, prop. 2, 23, 24, 25. pa. 35.....48.)
4. If any number of straight lines be given by position—and if to these straight lines, others be drawn, from a certain point, and at given angles—and if the rectangle contained by one of the lines so drawn, and a certain given line, together with the rectangle contained by another of the lines so drawn, and another given line—be equal to a third of the lines so drawn, and a third given line, and so on, that point will be situated in a straight line given by position.—(Simson, prop. 26, 27, 28, 29. pa. 49.....90.)
5. If, from a certain point, to two straight lines given by position, other straight lines be drawn, at given angles, and cutting off towards given points (in the given lines) segments, either having a given ratio to each other—or, containing a rectangle equal to a given space—or, so that the sum, or difference, of the [straight lined figures given in] species, described upon the said segments, be equal to a given space, that point will be situated in a straight line given by position.—(Simson, prop. 30, 31, 32, 33, 34. pa. 99....115.)

Book II.

1 & 2. If from two given points two straight lines be inflected to a third point, the difference of whose squares equal to a given space, the locus of their point of concurrence will be a straight line given in position: but if the lines be inflected be in a given ratio to each other, the locus of the point of concurrence may be either a straight line, or the circumference of a circle given in position; namely, a straight line—if the given ratio be that of equality—and a circle when the given ratio is not that of equality.—(R. Simson prop. 1 & 2. pa. 118....124.)

3. If there be a straight line given by position—and there be given in it a point—and if from this point there be drawn another (terminated) straight line—and from the end of this line there be drawn a perpendicular to a straight line given by position, and if the square of the line so drawn, be equal to the rectangle contained by the one that is given and the segment which it cuts off—either towards the same given point—or towards another given point in a straight line given by position—the extremity of the line which is drawn will touch the circumference of a circle given in position.—(Simson, prop. 3. pa. 125.....136.)

4. If from two given points straight lines be inflected to a third point, and the square of the one be greater than the square of the other—or in a given ratio to it—that point will touch the circumference of a circle given in position.—(Simson, p. 136.)

A. If from two given points straight lines be inflected to a third point, and if the square of one of them be given together with that to which the square of the other has a given ratio—that point will touch the circumference of a circle given in position.—(Simson, pa. 147.)

5. If from any number of given points there be drawn straight lines to one point—such, that the [straight figure given in] species described on them all—be equal to a given space—that point will touch the circumference of a circle given by position.—(Simson, pa. 159....182.)

6. If from two given points there be inflected two straight lines to a third point, and from the point of concurrence there be drawn a straight line parallel to another straight line given in position, which may cut off from the straight line given in position a segment adjacent to a given point—and if the straight lined figures given in species described on the inflected lines together, be equal to the rectangle contained by the given line, and the absciss, or segment cut off—the

point of concurrence will touch the circumference of a circle given by position.—(Simson, pa. 183.....193.)

7. If within a circle given in position, any point be given, and if through that point any straight line be drawn, and if on that line there be taken any point without the circle, so that the square of the straight line intercepted between those points, be equal to the rectangle contained by the whole line, and the absciss (or segment cut off) without the circle or to the square of the whole line—or segment—together with the rectangle contained by the two segments within the circle—the point without the circle will touch a straight line given in position.—(Simson, prop. 7, 8, 9, 10, 11, 12, 13, 14. pa. 199.....219.)

THE END.

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A

TREATISE

ON THE

EQUILIBRIUM OF ARCHES,

IN WHICH

THE THEORY IS DEMONSTRATED

UPON

FAMILIAR MATHEMATICAL PRINCIPLES.

THIRD EDITION.

By JOSEPH GWILT, ARCHITECT, F.S.A.

LONDON:

JOHN WEALE.

MDCCCXXXIX.

G. WOODFALL, ANGEL COURT, SKINNER STREET, LONDON.

INTRODUCTION.

AN arch is an artful arrangement of bricks, stones, or other materials, in a curvilinear form, which, by their mutual pressure and support, perform the office of a lintel, and carry or sustain superincumbent weights, the whole resting at its extremities upon piers or abutments.

The construction of an arch is one of the most important and curious operations in the science of architecture; it enables the artist to carry roads and canals over the widest valley, or most rapid river, to resist the greatest pressure, and at the same time to impart a light and beautiful character to his work.

An investigation of the pressures that take place amongst the materials whereof an arch is composed, and the method of arranging them so that the whole may remain at rest, are the subjects of the following work.

It does not appear by the remains of buildings still existing in Italy and other places, that the ancient architects in the construction of arches, were guided by equilibrial principles. They seem to have been unacquainted with any rules for resisting the displacement which the voussoirs are subject to by their mutual pressures, or for counteracting the lateral thrust of the whole arch against the piers. Experience and a certain mechanical perception seem alone to have guided them. We often find the piers of their bridges equal in thickness to nearly half the span of the arch. In other similar cases not a fourth part of the span is assigned to them. Had they proceeded upon fixed and decided rules or principles, this difference would not have existed.

Vitruvius is very particular with regard to the methods employed in building walls, bonding them, &c., and would, it may be presumed, have mentioned the principles upon which they proceeded in building their arches, had any then existed.

Probably, says Bossut, this part of the science of building, as well as the other constructive branches of it, was abandoned to the artificers, and the architect troubled himself about little besides the design and general distribution.

To the architects of the middle ages we are indebted for the noblest specimen of arched roofing. The skill employed in counteracting the thrusts, and balancing the pressure of the different arches in the cathedrals, and other remaining structures of our forefathers, exhibits a proof that the fraternity of freemasons had an accurate knowledge of this complex and difficult branch of the art.

It is not the object of this introduction to trace the history of the arch from its origin through its various modifications. The progress must necessarily have been slow, and marked by many degrees of improvement. Two stones inclined to, and meeting each other, required some care so to adjust that they should not push out the props or walls upon which they

rested. If, instead of letting the stones meet each other, a third stone be interposed, after the manner of a key-stone, we have the element of an arch. The conditions now change, and more adjustment will be necessary. Add a fourth stone, and still more science is requisite to prevent their fall.

It was comparatively late that the theory of arches attracted the notice of mathematicians. Dr. Hooke gave the hint, that the figure of a perfectly flexible cord or chain, suspended from two points, was the proper form for an arch.

Galileo considered the catenary as a parabolic curve, and John Bernouilli appears to have been the first who discovered its nature. Dr. Gregory (Phil. Trans. 1697) published an investigation of its properties, and observes that the inverted catenary is the best form for an arch on account of its lightness*. This is true so

* “*Catena in plano verticali, sed situ inverso, figuram servat nec decedit, adeoque arcum seu fornicem facit tenuissimum.*”

long as it is not pressed by an extraneous weight. It is not, however, capable of bearing a load on any part, much less of being filled up on the spandrels, which must be the case in practice. Other considerations must be involved before it can be fitted to receive a roadway or other weight, either upon its crown or haunches.

In 1695, La Hire, in his "*Traité de Mécanique*," established by the Theory of the Wedge, a method of determining the proportion in which the lengths of the voussoirs should increase as they approximate the springing.

The historian of the Academy at Paris, intimates, in 1704, that Parent following the same theory, but without proceeding further than mechanical delineation, gave the extrados of a semi-circular arch, together with its horizontal drift.

Couplet wrote two memoirs in the volumes of the French Academy for the years 1729 and 1730. He trod in the steps of La Hire and Parent, and made but little advances in the

scientific investigation of the subject. .He treats chiefly on the drift or shoot of semi-circular arches and the thickness of the piers, considering the arch-stones as infinitely smooth, and experiencing no resistance from friction.

In the volume of the Transactions of the French Academy for 1734, is a memoir by Bouguer, "*Sur les Lignes Courbes propres à former les Voutes en Dôme.*" In this is an analogy between cylindrical and dome vaulting, the one being supposed to be formed by the movement of a catenarian curve parallel to itself, and the other by the revolution of the same curve about its axis.

The wedge theory has since been followed by Belidor and others on the Continent, and the late ingenious Mr. Attwood in this country. It must not, however, be confounded with one of which a notice may be found in the Transactions of the Royal Irish Academy for the year 1789, in a memoir by Young on the Gothic arch, nor with the hint of one given by Dr. Gregory in

his *Mechanics*, second edition, 1807, the results of which, if carried through, would correspond with those in the following treatise.

Such was the state of the progress towards a perfect development of the true theory of arches, when our countryman Mr. Emerson in 1743 investigated the nature of a proper extrados of the different curves. Casting a new light upon this curious and useful subject, he exposed the fallacy of La Hire's wedge theory, and it is now completely exploded.

Dr. Hutton, in his "Principles of Bridges," and in the first volume of his tracts, pursued the subject on the principles laid down by Emerson. He added the method of finding an intrados to a given extrados, which indeed seems the principal addition he made to this branch of science.

Bossut, on the Continent, latterly republished with additions, in his "*Cours de Mathematique*," a tract on the subject; and in 1785, Mascheroni

published at Bergamo, a work intituled “*Nuove Ricerche sull’ Equilibrio delle Volte,*” in which there are some propositions upon the equilibrium of arches, but it principally relates to domes on circular, elliptical, and polygonal plans.

It now remains for the Author of the following treatise to state the reasons which first induced him to bring it before the public.

The works of Emerson and Hutton are elegant, as might be expected from the pens of such excellent mathematicians, but they are nevertheless of little if of any service to a profession, whose province it is to practise that science, whereof a just knowledge of the theory of arches forms but a small proportion. The fluxionary calculus is used by both authors; this is little understood by the profession in general, nor indeed is it absolutely necessary that it should be. The analytical process to any other person than a mathematician, is not so striking as a

demonstration by means of lines, where the operations constantly strike the sense*.

It therefore appeared to the Author, that a short Treatise, founded on the theory of Emerson, wherein the demonstrations did not require an acquaintance with the higher branches of the mathematics, would be profitable to the students of architecture, and perhaps even to the professors. It is not, however, meant to be inferred that the architects of this country are destitute of mathematical acquirements; but it must be recollected, that many sciences are connected

* Rousseau, speaking of the application of algebra to geometry (Confessions, Liv. 6,) says, “ Je n’ai jamais été assez loin pour sentir l’application de l’algebre à la geometrie. Je n’aimois point cette manière d’operer sans voir ce qu’on fait; et il me sembloit, que resoudre un problème de geometrie par les equations c’etoit jouer un air en tournant une manivelle.— Ce n’etoit pas que je n’eusse un grand gout pour l’algebre en n’y considerant que la quantité abstraite; mais appliqué à l’etendue, je voulois voir l’operation sur les lignes, autrement je n’y comprenois plus rien.”

with their profession, and that a perfect knowledge of each cannot be expected of them *. Neither is it intended to disparage the analytical art, the importance and utility of which, in investigations similar to those above alluded to, is obvious.

Under the foregoing circumstances, the Author flattered himself that the following pages would meet with that indulgence from mathematicians to which, from the nature of his avocations in business, he conceived himself fairly entitled. Such has been the case; and in the second edition, in which many important alterations and

* Vitruvius, after enumerating the many qualifications of an architect, says, “that he need not be singularly excellent in the sciences, though he should not be entirely without a knowledge of them; for in such a number of different things, it is not possible to attain to singular perfection in each.”—“Nor is it architects alone who cannot arrive at this pitch of eminence in all literature; for even some of those who have professed a single art, have not been able to obtain the highest degree of reputation therein.” Book I. Chap. I.

additions were made, he trusts that his endeavours to elucidate and render more familiar an interesting and highly important branch of science, will be received by the profession of which he is a member, as a token of respect and esteem.

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A

TREATISE

ON THE

EQUILIBRIUM OF ARCHES.

SECTION I.

*Of the general Laws of Motion and the Composition
and Resolution of Forces.*

DEFINITION I.

WHATEVER changes or tends to change the condition of a body, with respect to rest or motion, is called force: thus, pressure, impact, gravity, &c. are called forces.

DEFINITION II.

Momentum signifies the quantity of motion in a body, and is measured by the velocity and

B

quantity of matter jointly : thus, if the quantity of matter in A is represented by 10, and that in B by 12, and their velocities by 15 and 11 respectively, then the ratio of their momenta is as 10 multiplied by 15 is to 12 multiplied by 11 or as 150 to 132.

DEFINITION III.

Mechanical action is that of a moving force, and its effect is measured by the momentum generated in a given time.

AXIOMS.

1. Every body will continue in its existing state, either of rest or uniform rectilinear motion, until acted upon by some force tending to change its state.

This law is conformable to our ideas of matter and motion, for where no change of condition in a body is supposed, no power is supposed to act; and when a change is supposed, then also some power is supposed to act. The law is usually illustrated by this familiar circumstance, that when a man is standing in any vehicle, if the vehicle suddenly change its state from rest to

motion, or the contrary, the man is in danger of being thrown from his actual position.

2. Every motion or change of motion in a body, is proportional to the force exerted to produce it, and is in the direction of the right line in which such force acts.

Though this law also depends upon our ideas of matter and motion, yet its truth is also inferred from experiments on the collision of bodies; and though some cases occur in which the velocities generated by forces are not, simply, proportional to those forces, it is supposed that, in such cases, only parts of the forces are effective in producing the observed velocities.

3. Action and re-action between any two bodies are always equal and opposite in direction.

By action and re-action equal changes of condition are produced in bodies acting on each other: for, let A be a simply hard body, as a ball moving with a given velocity, and let it impinge upon another equal ball at rest; the two balls will, after impact, move together with half the velocity which A had before impact. It is evident, therefore, that A communicates to B a velocity equal to half its own velocity, and

is, by the re-action of B, deprived of just so much velocity as it has imparted to B. Thus the momentum which is measured as above, remains unaltered.

Having defined what is meant by a force, and shewn the most general laws of its action, we proceed to state the circumstances attending the joint action of two or more forces upon a body subject to their influence. Two forces may act upon a body in three different ways; they may act in the same, or in opposite directions, or they may act obliquely to each other.

If two magnitudes, as two lines or two numbers, be taken to represent two forces, the sum of those magnitudes must represent a force which is equivalent to the two given forces when they act in the same direction; and the difference of the magnitudes must represent a force equivalent to the forces when they act in opposite directions: so that in this case, when the forces are equal, the difference of the magnitudes being nothing, shews that an equilibrium subsists, or that the forces destroy each other's effects.

But if the forces act obliquely on a body, it is evident that there must exist one single force capable of producing the same effect upon the

body as the given forces; and this is what we purpose next to determine.

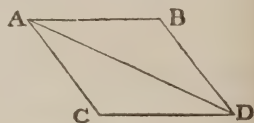
This proposition and its converse, which form the subject of the composition and resolution of forces, admit of demonstration, from the consideration of the forces themselves, without requiring the idea of motion to enter into the investigation; but the great intricacy of such demonstration * renders it preferable to avail ourselves of the second law of motion; viz., that the motion produced in any body is proportional to the acting force; in which case we may employ the motions, or spaces described, to represent the forces themselves; and therefore we may suppose the given forces to be such that they would cause the body to move with similar motions through certain spaces respectively in the same time, and we shall then only have to determine the space described in the same time by the given forces acting together. This being understood, we proceed to state the following propositions.

* See Robinson's *Mech. Philos.*, or Gregory's *Mechanics*.

PROPOSITION I.

If a body at A be urged by two similar forces, so that they would separately cause it to move along the adjacent sides A B, A C, in an equal time : then will those forces, communicated at the same instant, cause it to pass through the diagonal A D, of the parallelogram A B C D, in the same time.

For the force (Axiom 2.) along the side A B can make no alteration in the body's approach to C D, to which A B is parallel, since it will arrive at C D



in the same time that it would have done had no motion been communicated in the direction A B. In the same manner it will arrive in B D in the same time that it would have done had no motion been communicated to it in the direction A C. It will therefore be found at the intersection D of the sides C D and B D. And as the forces are supposed similar, the body will move with a rectilinear motion, and will consequently pass along the diagonal A D of the parallelogram A B C D.

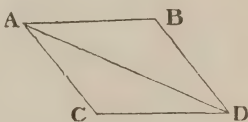
COROLLARY.

If two sides of a triangle represent the spaces over which two forces would separately carry a body in an equal time, the third side will represent the space over which these forces acting conjointly would urge it uniformly in that time. For if AC and CD be the two sides; by completing the parallelogram AD , the third side will be the diagonal thereof.

PROPOSITION II.

If AB , AC , the adjacent sides of the parallelogram $ABCD$, represent the quantities of two forces acting upon a body at A , as also the directions of those forces, then will the diagonal AD represent a force equivalent to them.

By the preceding proposition, AD is the space described when the body is urged from A by two similar forces at the same instant; and as AB , AC , and AD , represent the spaces passed over in equal times, they also represent the momenta; and as the spaces or the momenta may



be taken to represent the forces themselves, for the reason before given, consequently the forces AB , AC , compounded or acting on A at the same instant, exert a force whose equivalent is AD .

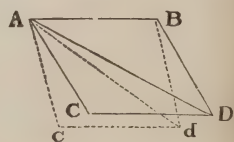
COROLLARY I.

If two sides of a triangle represent in quantity and direction two forces acting on any body, then will the third side represent a force equivalent to them both, acting conjointly.

COROLLARY II.

As the angle in which two forces act is diminished, the compound force is increased.

Let AB , AC , represent two forces acting in the lesser angle CAB .
and AB , Ad , two other forces respectively equal to the former, and acting in the greater angle dAB .



It is manifest that the angle ABd is less than the angle ABD , and the sides BD , Bd , being equal, and AB common to both; therefore (Euc. Prop. 24, B. 1.) the base AD , of the triangle ABD , will be greater than the base

A d, and A D is the equivalent of the two forces acting in the lesser angle.

COROLLARY III.

Two forces acting in the same direction, produce the greatest effect.

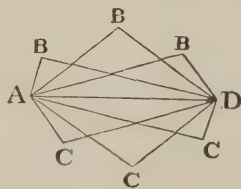
COROLLARY IV.

A body cannot be kept at rest by two forces, unless they are equal and in opposite directions.

PROPOSITION III.

A single direct force A D, may be resolved into any number of oblique forces.

For the direct force A D may be resolved into as many pairs of forces as there may be triangles described upon it, as A B D, &c. &c. or parallelograms A B C D, &c. &c.; to all the pairs of which separately, A D will, by the last proposition, be equivalent.



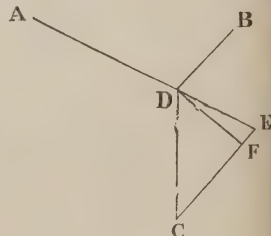
COROLLARY.

If two forces are together equivalent to $A D$, and $B D$ be one of them, $A B$ must be the other.

PROPOSITION IV.

If three forces A, B, C , acting together be in equilibrio, they are proportional to the three sides $D E, E C, C D$, of a triangle drawn parallel or perpendicular to the directions of those forces.

Let $C D, A D$, and $B D$, represent the direction of the forces. Produce $A D$ indefinitely, and at C parallel to $D B$, draw $C E$ meeting $A D$ produced in E . Then $D C, E C$, and $E D$, represent the three forces.



For take $D C$ to represent the force in that direction; then if $E C$ do not represent the force in the direction $D B$, let $F C$ represent it, and join $F D$. Then (Prop. 2. Cor. 1.) if $D C$,

CF , represent the quantities and directions of any two forces, FD will represent the quantity and direction of their equivalent. But as DC , CF , represent the forces in CD , BD , and as the two latter are kept in equilibrio by the force in AD , and that by hypothesis, DF is the equivalent of those latter; it follows that DF is equal and opposite to AD , or AD , DF are in the same straight line. Hence the straight lines ADE , ADF , have a common segment, which is impossible (Euc. Cor. Prop. 11. B. 1.) Therefore EC represents the force in the direction DB , and consequently ED represents the third force; and any three sides of a triangle drawn parallel to the directions of three forces AD , DB , DC , acting together, are proportional to the sides of the triangle DEC , and consequently proportional to the three forces.

Again, since if three lines be drawn perpendicular to the three sides of a triangle, they will form a triangle similar to that given; it follows, that, if three forces are in equilibrio about a point, they will be proportional to the sides of a triangle drawn perpendicular to the directions of those forces.

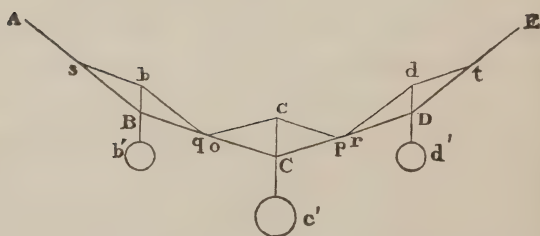
COROLLARY I.

Because the three sides of a triangle are proportional to the sines of their opposite angles, therefore three forces, when in equilibrio, are proportional to the sines of the angles made by their lines of direction.

COROLLARY II.

If one of the forces, as C, be a weight sustained by the two strings D A, D B, the force or tension of the string A D, is to C, or the tension of the string D C, as D E to D C, and the tension of B D is to C, as C E to C D.

LEMMA.



Let A B C D E be a thread stretched throughout in the direction of the different parts of its

length AB , BC , CD , DE , (the ends A , E , being fixed,) by means of weights suspended from the angles B , C , D , acting vertically; and let the weight c' be represented by the vertical Cc . Consider such vertical as the diagonal of a parallelogram, whose sides are parallel to BC and CD , and complete the same. Make Bq and rD respectively equal to oC and Cp , and from the points q and r , or the indefinite verticals bB , dD , complete parallelograms, whose sides are parallel to AB , BC , and CD , DE . Then Bb , Dd , will represent the weights which must be suspended at the points B and D to preserve the figure, and Bs , Bq , Dr , Dt , will represent the contractile forces or tensions at B and D in the directions of those lines respectively. The truth of this lemma is evident from Prop. 2. For Bb , Cc , Dd are forces equivalent to those represented by Bs , Bq , Co , Cp , &c. respectively, and because $Bq = oC$, $Cp = rD$, &c. the tensions at B and C , C and D are equal in opposite directions. Therefore the figure subsists in equilibrio.

If the figure be inverted, and considered as consisting of props instead of strings, the same equilibrium will obtain, the weights being

brought to act in a vertical direction on the angles.

Props inverted act as strings, and the converse; the former suffering compression, and the latter tension.

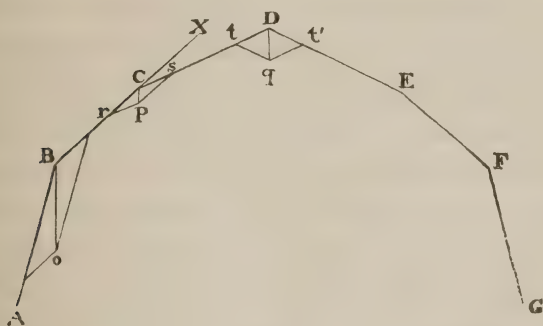
If A B C D E were a thread or chain attached to immoveable points at A and E, and left, without any accidental weights, to the action of gravity, it would assume the form of that particular curve which has been called a simple catenary; and if a system of indefinitely small and equal bodies were arranged in the same form, but in an inverted position, it would also stand in equilibrio, and such a figure would exhibit what might be called an arch of equilibration; such arch, however, would be of no practical use, because it could not be loaded. The case would be different if the catenary were of the kind called complex, as we shall presently see.

PROPOSITION V.

PROBLEM.

The lines A B, B C, C D, are inclined to each other in angles given or known, and the weight

placed over the angle D is also given; required the weights which must be placed on the other angles B and C, so that the whole assemblage may remain in equilibrio, A and G being fixed.



Describe parallelograms, and proceed as in the foregoing lemma, D q being the weight, or that which represents the weight at the angle D.

Then (Prop. 4. Corol. 1.) those forces which balance the lines at the angles B, C, D, &c. are as the sines of the angles of a triangle, formed by their lines of direction, and in the parallelogram C r p s, C p expresses the force at C, acting vertically, and C r, C s, the equivalent forces into which it is resolved.

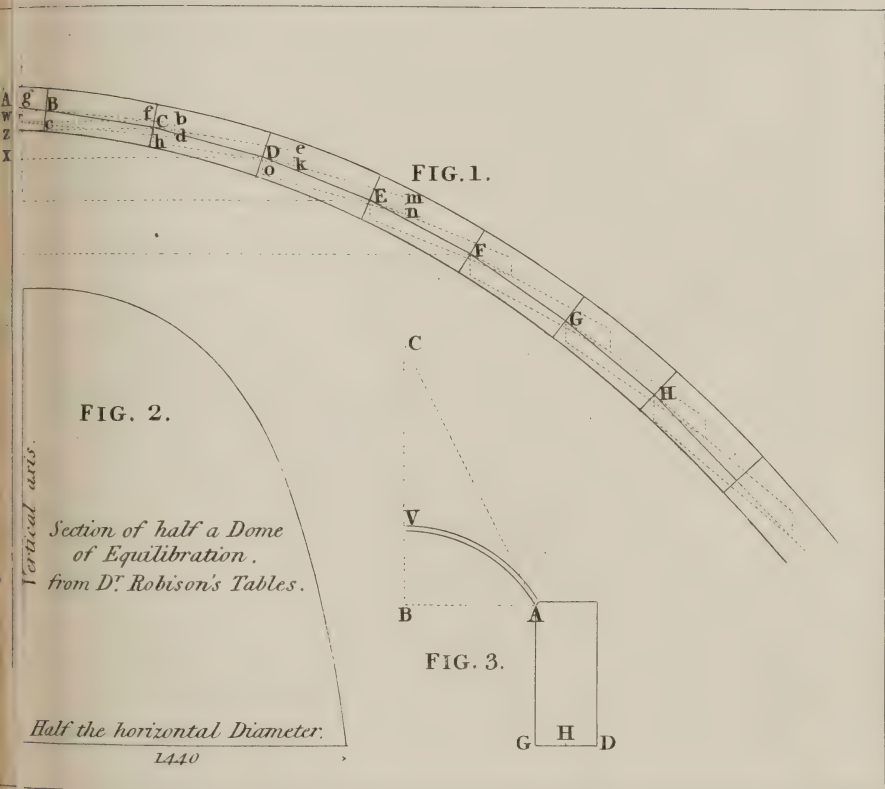
1. Produce r C to X, then will $\angle s C X =$

$\angle p r C$, and as $\text{sine } \angle r C s = \text{sine } \angle s C X$, it is also equal to $\text{sine } \angle p r C$, to which $C p$ is opposite.

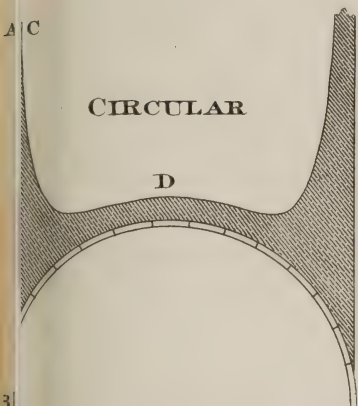
2. $r p$ is opposite to $\angle r C p$, and is equal to $C s$ or $t D$.

3. The angle $r p C = \angle p C s$ and $r C$ is opposite to $\angle r p C$, whose sine is consequently equal to the sine of $\angle p C s$.

Therefore the three forces $C p$, $C s$, and $C r$, are respectively proportional to the sines of the angles $p r C$, $r C p$, and $p C s$. From whence may be deduced, that the weight on any angle is directly as the sine of that angle, and reciprocally as the sine of the two parts into which it is divided by a vertical line, as follows. The force $C r : \text{force } C s :: \text{sine } \angle p C s : \text{sine } \angle r C p$. But $s \angle p C s = s \angle q D t$, since those angles are supplements of each other. Then by the nature of proportion, the force $C r : \text{force } C s :: \frac{1}{s \angle r C p} : \frac{1}{s \angle q D t}$. In the same manner it may be proved, that the force $t D : \text{force } D t' :: \frac{1}{s \angle q D t} : \frac{1}{s \angle q D t'}$ and so on. Whence it is evident that the forces $C r$ and $D t$ vary as $\frac{1}{s \angle r C p}$ and $\frac{1}{s \angle q D t}$.



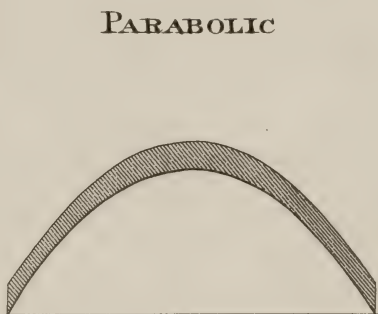
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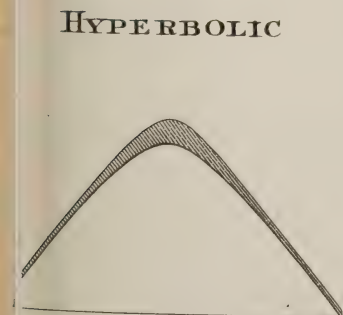
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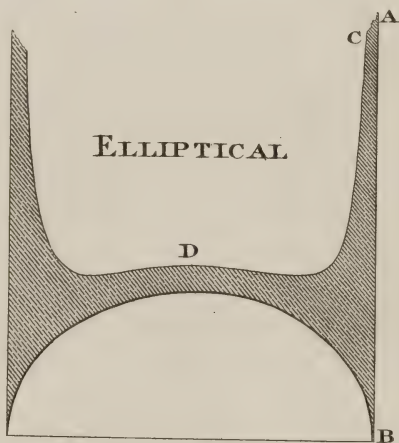
When the intrados is an entire
micircle AB is infinite being an
asymptote to the extrados CD.



PARABOLIC



HYPERBOLIC



ELLIPTICAL

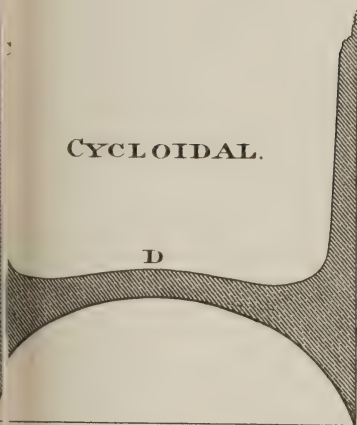
D

When the intrados is an entire
Semi ellipsis AB is infinite being an
asymptote to the extrados CD.

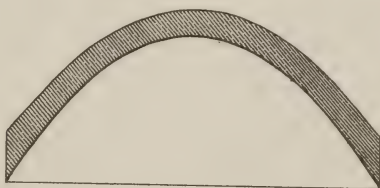
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CYCLOIDAL.

D



CATENAREAN.



AB is infinite and is an asymptote to CD the extrados.

FIG. A.

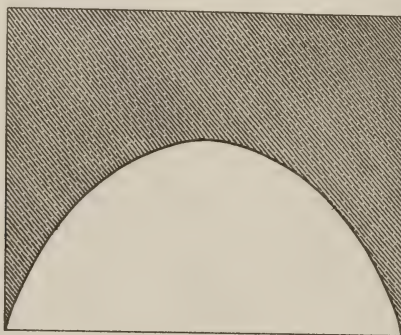
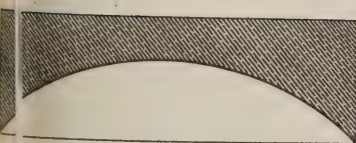


FIG. B.

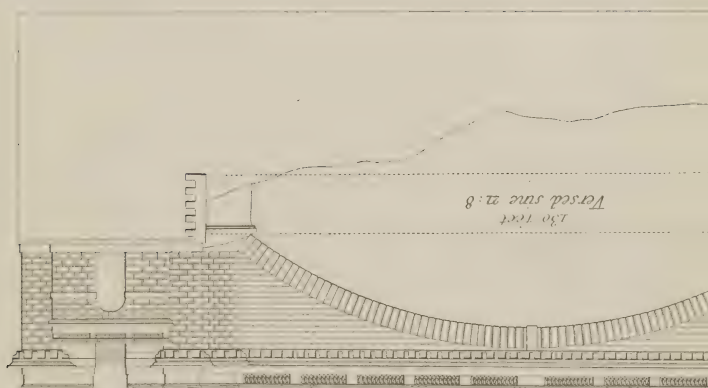


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S

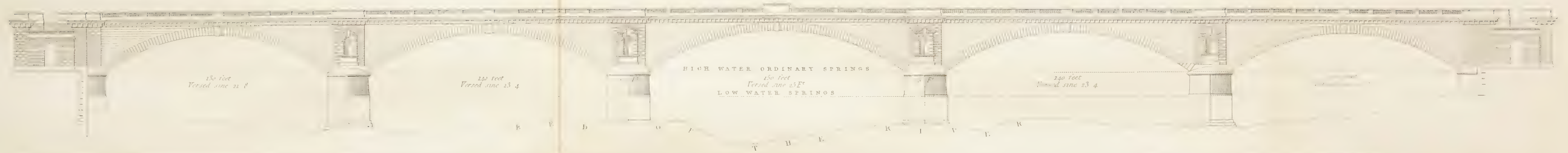
R



Wassel. 21:8
130 feet

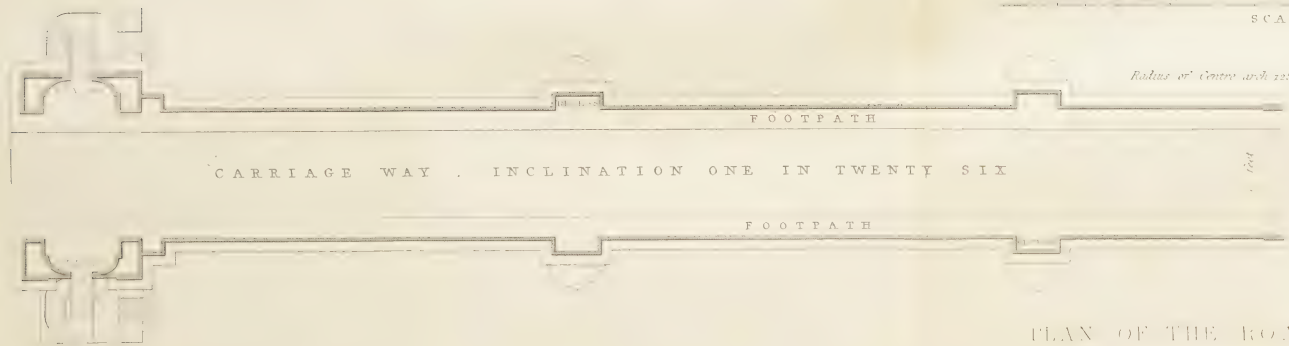
DESIGN
FOR THE NEW LONDON BRIDGE. MDCCCXXII.

PROPOSED BY JOSEPH GWILT, ARCHITECT, F.S.A.



ELEVATION

50 200 250 Feet
SCALE.



PLAN OF THE ROADWAY AND PIERS.

Published by Peckham & Wale, High Street, Birmingham 1842.
Printed by J. B. B. 1842.

In figures, let . . . $Aq = 1.00$

$$AB = 40.00$$

$$B'B' = 100.00$$

$$AO = 10.00$$

$$\text{Consequently } AR = 51.25$$

Now (see page 25).

$$\frac{(\text{sec. at vertex})^3}{AR} : Aq :: \frac{(\text{sec. } \angle TCO)^3}{AR} : Cp.$$

Then by the nature of the circle, the radius of curvature is constant, and we shall have $\angle TCO^* = 36^\circ 24'$, so that the analogy becomes,

* To find the angles which the line CT forms with CO, we must proceed as follows :

When the curve is circular (see figure to the proposition),

$$CO = \sqrt{2AR \times OA - OA^2}$$

$$TO = \frac{OC^2}{OR}.$$

Then by plane trigonometry,

Since $\angle COT$ is a right angle,

$$CO : \text{radius} :: TO : \text{tang. } \angle TCO.$$

When the curve is parabolic (see figure to prop. 4).

$$CO = \sqrt{\frac{AO \times B'B^2}{AB}}$$

$$OT = 2AO$$

Then by trigonometry we have $\angle TCO$ as in last.

(sec. $0^{\circ} 00'$)³ or radius : 1 :: (sec. $36^{\circ} 24'$)³ : C p = 1.9178.

Then by taking different points in the curve, and proceeding as above, we shall have the different heights of the superincumbent wall at those points as required.

At the vertex A the secant is equal to the

When the curve is elliptical (see figure to prop. 2).

$$CO = \sqrt{\frac{B'B^2 \times BA^2 - BO^2}{BA^2}}$$

$$OT = \frac{BA^2}{BO} - BO.$$

Whence we have $\angle TCO$ as before.

When the curve is hyperbolic (see figure to prop. 5).

$$CO = \sqrt{\frac{B'B^2 \times BO^2 - BA^2}{AB^2}}$$

$$OT = BO - \frac{BA^2}{BO}.$$

Whence we have $\angle TCO$ as before.

When the curve is cycloidal (see figure to prop. 3).

By the properties of the cycloid (prop. 4, sect. 2),

$$\angle TCO = \angle ACO.$$

Where AB is the diameter of the generating circle.

$$\text{And } OC' = \sqrt{BO \times OA}$$

And by plane trigonometry,

$$OC' : \text{radius} :: AO : \text{tang. } \angle OC'A = \text{tang. } \angle TCO.$$

From prop. 5 in the last section, the reader may also find the $\angle TCO$ in the catenarian curve.

radius, and may be represented by CO ; and the secant of the curve's inclination to the horizon at C , may be represented by TC , we shall therefore have the following analogy :

$$CO^3 : Aq :: TC^3 : Cp.$$

But because of the similarity of the triangles ORC , and TCO , OR and CR will be constantly proportional to CO and CT .

Consequently

$$OR^3 : Aq :: CR^3 (=AR^3) : Cp;$$

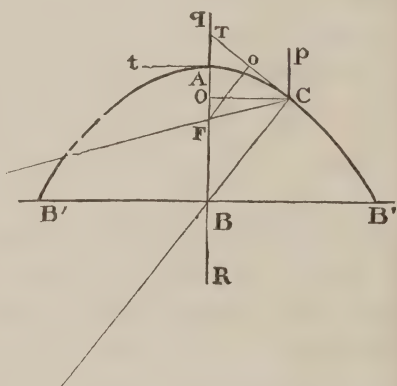
$$\text{that is } \frac{Aq \times AR^3}{OR^3} = Cp,$$

which will be a general expression for the height Cp at any point C of the curve; and is the same as Emerson makes it (see his *Doctrine of Fluxions*, London, 1743, page 248).

PROPOSITION II.

To determine the height of the extrados above every point of a parabolic arch.

Let $B' A B'$ be a parabolic arch, whose span is $B' B'$, and height $A B$.



Through F the focus of the parabola, draw CF , and from F let fall

the perpendicular Fo upon TC , then will the triangle FoC be similar to the triangle TOC ; for the alternate angles OTC and TCp are equal, and the angle oCF is equal to the angle TCp . Consequently $\angle OTC = \angle oCF$.

The angle TOC is also equal to the angle FoC , being both right angles.

Consequently Fo may represent OC , the secant at the vertex, and FC may represent TC , the secant at the point C .

Therefore $F o^3$ and $F C^3$, are the cubes of the secants at those two points.

Again, the radius of curvature at C varies as $\frac{F C^2}{F o}$ (see Vince's Conic Sections, cor. to prop. 9), and when C arrives at A , $F C$ and $F o$ become each equal to $F A$.

$$\text{Hence } \frac{F C^2}{F o} = F A.$$

(See same work, prop. 8.)

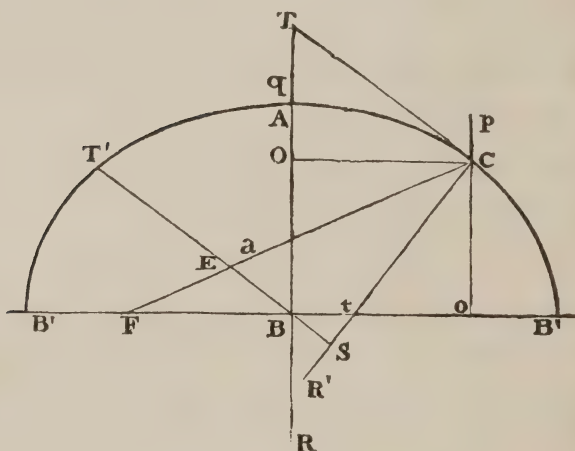
$$\text{Therefore } \frac{\frac{F o^3}{F o^2}}{\frac{F C}{F o}} : A q :: \frac{\frac{F C^3}{F C^2}}{\frac{F o}{F o}} : C p;$$

$$\text{Or, } \frac{F o^3}{F o^3} : A q :: \frac{F C^3}{F C^3} : C p, \text{ a ratio of equality.}$$

Therefore $C p$ is constantly equal to $A q$.

PROPOSITION III.

To determine the height of the extrados, above every point of an elliptical arch.



Let $B' A B'$ be a semi-elliptical arch, whose span is $B' B'$, and height $A B$.

Draw $B T'$ parallel to the tangent $C T$, and from the point C let fall $C o$ perpendicular to $B' B'$, and $C S$ perpendicular to $T' B$ produced, if necessary, cutting $B' B'$ in t .

Then because the angles T C S and O C o are right angles, and O C t is common to both,

take away the angle $O C t$ from each, and the angle $T C O$ is equal to the angle $t C o$. Also the angle $C O T = \text{angle } C o t$ being both right angles; therefore the triangle $C O T$ will be similar to the triangle $C o t$.

Consequently $C o = B O$ may represent the radius or secant at the vertex, and $C t = \frac{A B^2}{C S}$, (see Vince's Conic Sections, prop. 15) the secant at the point C .

Therefore $B O^3$ and $\frac{A B^6}{C S^3} =$ the cubes of the secants at those two points.

From C to F , one of the foci of the ellipse, draw $C F$, cutting $B T'$ in E , and cut off $C a$ equal to the chord of curvature passing through the focus (prop. 2, sect. 2), then will $C S$ be to $C E$, as $\frac{1}{2} C a$ is to the radius of curvature (see Vince's Conic Sections, prop. 21).

Now $C E$ is always equal to $B' B$, and when C arrives at A , $C S$ becomes $A B$, also $T' B$ coincides with $B' B$, and $\frac{1}{2} C a$ or $\frac{T' B^2}{B B'}$, becomes equal thereto.

Therefore $A B : B' B :: B B' : \frac{B' B^2}{A B}$ the radius of curvature at A .

$$\text{And } C S : B' B :: \frac{1}{2} C a \left(= \frac{B T^{1/2}}{B B'} \right) : \frac{B T^{1/2}}{C S}$$

the radius of curvature at C (see above work, prop. 21).

$$\text{But } \frac{B' B^2 \times A B^2}{C S^2} = T' B^2 \text{ (same work, prop. 11.}$$

cor. 2) ;

$$\text{therefore } \frac{T' B^2}{C S} = \frac{B' B^2 \times A B^2}{C S^3}.$$

Consequently,

$$\frac{\frac{B O^3}{B' B^2} : A q :: \frac{\frac{A B^6}{C S^3}}{B' B^2 \times A B^2}}{\frac{A B}{C S^3}} : C p,$$

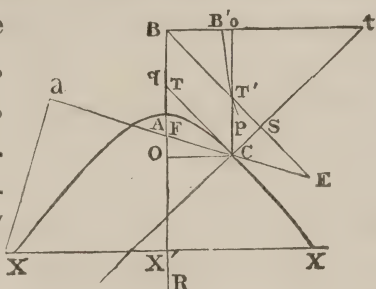
$$\text{Or, } B O^3 : A q :: A B^3 : C p;$$

$$\text{That is, } C p = \frac{A q \times A B^3}{B O^3}.$$

PROPOSITION IV.

To determine the height of the extrados above every point of an hyperbolic arch.

Let $X A X$ be an hyperbolic arch, whose span is XX , and height $A X'$. $A B$ being the semi-transverse, and $B B'$ the semiconjugate axis.



From C draw Ct perpendicular to CT , and produce BB' to meet Ct in t , and draw Co parallel to BO , then will the triangles TCO and oCt be similar, for the angles OCO and TCt are equal, being both right angles; take away from each the common angle TCo , there will remain the angle TCO equal to the angle oCt , and the angles TOC and toC are both right angles. Therefore oC which is equal to BO , may represent the secant at the vertex, and Ct the secant at C . Now as in the ellipse,

$$Ct = \frac{AB^2}{CS}.$$

Therefore BO^3 and $\frac{AB^6}{CS^3}$ will be the cubes of the secants at those points.

Again through F, one of the foci of the curve draw FC, and produce it towards E and a, and cut off Ca = the chord of curvature. Also from B draw BE parallel to TC, cutting Ct in S.

Then (as in the ellipse) $CS : CE :: \frac{1}{2} Ca : \text{radius of curvature}$.

Now $CE = AB$ always, and when C arrives at A, CS also becomes equal to AB, and the half chord of curvature becomes equal to $\frac{B'B^2}{AB}$.

Therefore $AB : AB :: \frac{B'B^2}{AB} : \frac{B'B^2}{AB}$ the radius of curvature at A.

And $CS : AB :: \frac{1}{2} Ca \left(= \frac{BT'^2}{AB} \right) : \frac{BT'^2}{CS}$
the radius of curvature at C.

$$\text{But } BT'^2 = \frac{AB^2 \times B'B^2}{CS^2}.$$

$$\text{Therefore } \frac{AB^2 \times B'B^2}{CS^3} = \frac{BT'^2}{CS}.$$

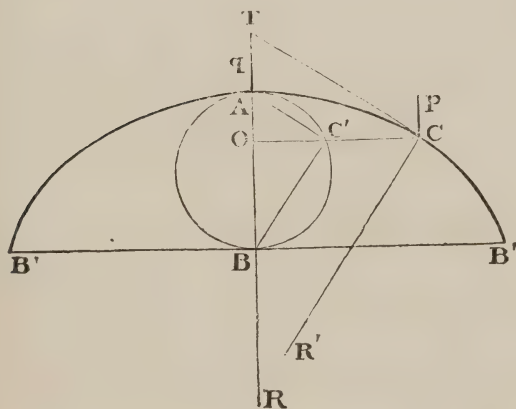
$$\text{Consequently } \frac{\frac{B O^3}{B' B^2}}{\frac{A B}{A B}} : A q :: \frac{\frac{A B^6}{C S^3}}{\frac{A B^2 \times B' B^2}{C S^3}} : C p$$

$$\text{Or } B O^3 : A q :: A B^3 : C p;$$

$$\text{That is } C p = \frac{A q \times A B^3}{B O^3}.$$

PROPOSITION V.

To determine the height of the extrados above every point of a cycloidal arch.



Let $B' A B'$ be a cycloidal arch, whose span is $B' B'$, and height $A B$.

Then

$$\frac{(\text{sec. at vertex})^3}{A R} : A q :: \frac{(\text{sec. } \angle T C O)^3}{C R'} : C p;$$

$$\text{Or } \frac{O C^3}{A R} : A q :: \frac{T C^3}{C R'} : C p.$$

Draw $A C'$ parallel to $C T$, then by the nature of the cycloid, because $T C$ is always parallel to $A C$, the triangle $T C O$ will be similar to the triangle $A C' O$, and consequently, to the triangle $C' B O$, and their sides will be respectively proportional to each other; that is,

$$O C : T C :: B O : B C'.$$

And as the radii of curvature at A and C are respectively equal to $2 A B$ and $2 B C'$, they are to one another as $A B$ to $B C'$; that is as $B C'$ to $B O$. Consequently substituting these terms in the last proportion, we shall have

$$\frac{B O^3}{B C'} : A q :: \frac{B C'^3}{B O} : C p,$$

or by dividing the antecedent terms by $\frac{B O}{B C'}$,

$$\text{we have } B O^2 : A q :: \frac{B C'^4}{B O^2} : C p.$$

$$\text{But } \frac{B C'^4}{B O^2} = B A^{2*},$$

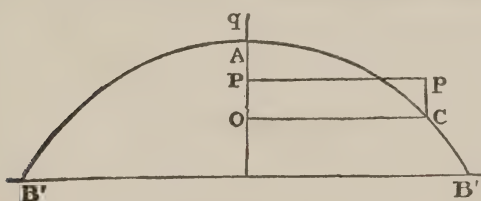
* For (Euclid Cor. Prop. 8. B. 6,) $B C'$ is a mean proportional between $B A$ and $B O$, whence

Therefore $BO^2 : Aq :: BA^2 : Cp$, that is,

$$Cp = \frac{Aq \times BA^2}{BO^2}.$$

PROPOSITION VI.

To determine the height of the Extrados above every point of a Catenarian Arch.



The same analogy will obtain as in the other Propositions, the radius of curvature, &c. being found as directed in Sect. 2, Prop. 5.

Or let the tension of the curve at the vertex be called X .

Then (by the nature of the catenary) X will be constantly to AO ,

As $X - Aq$ to qP .

That is $X : AO :: X - Aq : Pq$.

$\frac{BC'^2}{BO} = AB$; and squaring both sides, we have

$$\frac{BC'^4}{BO^2} = AB^2.$$

$$\text{Whence } P q = \frac{A O \times X - A O \times A q}{X}.$$

$$\text{But } C p = A O + A q - P q.$$

Therefore

$$C p = A O + A q - \frac{A O \times X - A O \times A q}{X}, \text{ which}$$

$$\text{being reduced becomes } C p = \frac{A q \times (X + A O)}{X}.$$

OBSERVATIONS.

The subjoined table will shew at one view the height pC of the superincumbent wall, over any point C of the intrados; and by referring to the figures in plates 1 and 2, we shall see the general form of the extrados to each of the curves that have been treated of. In Emerson's Fluxions the reader will find, besides the curves already mentioned, the mode of determining the extrados to the logarithmic curve and cissoid; but as these curves would be seldom, if ever used in practice, it has been considered unnecessary to give them a place in this Treatise, which is designed to instruct the practical artist.

Curve.	Value of Cp at any point C .	
Circle.	$\frac{Aq \times AR^3}{OR^3}$ or	{ The height at the crown multiplied by the cube of the radius, and the product divided by the cube of the vertical height of the point C from the horizontal diameter.
Parabola.	$pC = Aq$ or	{ The vertical height of the extrados every where equal.
Ellipsis.	$\frac{Aq \times AB^3}{OB^3}$ or	{ The height at the crown multiplied by the cube of the semiconjugate axis, and the product divided by the cube of the vertical height of the point from the transverse axis.
Hyperbola.	$\frac{Aq \times AB^3}{(BA + OA)^3 \text{ or } BO^3}$ or	{ The height at the crown multiplied by the cube of the semitransverse axis, and the product divided by the cube (of the semitransverse axis added to the abscissa).
Cycloid.	$\frac{Aq \times AB^2}{(AB - AO)^2}$ or	{ The height at the crown multiplied by the square of the diameter of the generating circle, and the product divided by the square (of the diameter aforesaid less the abscissa).
Catenary.	$\frac{Aq \times (X + AO)}{X}$ or	{ The height at the crown multiplied into the sum of the tension at the vertex and the abscissa, and the product divided by the tension.

Thus it appears that all arches which spring perpendicularly, or whose secants at the springing are infinite, as the semi-circle, semi-ellipsis, and cycloid, will require to be loaded with an infinite weight over that point.

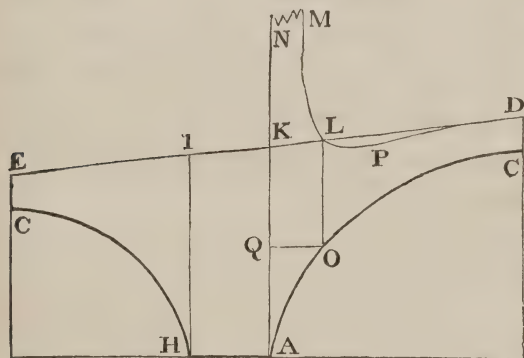
The extrados of a parabola will be another parabola, since pC is every where equal. And in the hyperbola the extrados continually approaches the intrados; but the scantiness of the haunches of these two curves, independent of their extreme meagre aspect, renders them unfit for the purposes of a bridge. Where great weights are required to be discharged (under particular circumstances) from the weakest parts of an edifice, as is often necessary in warehouses, and sometimes under apertures inverted, in order to bring an equal and uniform pressure on the foundations, the parabolic arch may with great propriety be introduced.

The catenarian curve will seldom be admissible, more especially in bridges; for where an horizontal roadway is required, the height at the crown must be very great. An arch of this kind of $31\frac{3}{10}$ feet span, and $15\frac{9}{10}$ feet rise, with an horizontal extrados, must be ten feet high at the crown (see fig. A, plate 2).

In general, however, by using segments of the ellipsis, cycloid, and circle, we may obtain

convenient roadways or extradoses, little differing from the conditions of the theory. In a segment of a circle, for instance, of 90 degrees, if the height at the crown be to its radius as 1 to $6\frac{1}{4}$, that is, about one-ninth of the span, the roadway will be nearly horizontal (see fig. B, plate 2).

The numerous instances in which arches spring perpendicularly, without the infinite load which this theory requires over the springing point being given to them, seem to induce doubts of its truth; let us therefore shortly consider the apparent variance.



Take for instance two semi-arches A C, C H, of a bridge (segments of circles); E I K L D is the extrados likely to be applied in practice,

in which the part K L D is the actual extrados over A O C; but the extrados which the theory requires is M L P D. Now D P L it will be seen by inspection nearly coincides with it as far as L; but the part of the extrados over A O appears deficient in the quantum of weight with which it ought to be loaded to obtain an equilibrium. But I H A O L, connected as it is by the bond of the stones of which it is composed, and the cement which unites them, may be considered as a solid mass of stone acting by its absolute gravity in a vertical direction upon A O Q, the part of the arch which requires the weight K L M N, and therefore may be considered as exerting a pressure equal to that of the infinite loading which theory requires. Thus in most cases the mode of constructing arches is not very far out of the conditions of the theory.

SECTION IV.

Of the Method of finding an Intrados to any given Extrados.

If the extrados or roadway over an arch be considered polygonal, (as we considered the intrados in the first Section,) and the direction of the first line of the intrados be given, we shall have an easy and tolerably correct method of finding the rest of the lines which compose the intrados, so as to equilibrate with the extrados; for the sides of both may be increased in number, so as to coalesce with the curves of which these lines are the chords (see pages 22 and 23, Sect. I.)

Let $F 1, 1 m, m n, n o', o' p', p' q'$, (fig. 1, plate 3) be the sides of an extrados, and let $C D$ be the direction of the first side of the intrados, cutting a vertical let fall from p' in C , $D q'$ being the height at the vertex D .

Make the angles $q D t, D q t$, equal to the angle $q D C$, make $D q$ equal to $q' D$, and

draw the lines Dt , qt , intersecting each other in t .

Produce (if necessary) CD indefinitely towards s , and in it take Cs equal to qt , and in the vertical $p'C$ produced, make Cp equal $p'C$, and join ps .

Through C parallel to ps draw CB , cutting a vertical from o' in B , and produce it towards r , then making Br equal to ps , and joining or , proceed as above for the heights under the points of the extrados n, m, l ; or is the direction of the side under no' .

Through all the points thus found, as also those of the extrados, bend a flexible ruler and draw the curve lines, which will be similar to qG and DE .

The operations for finding an intrados to any other extrados, will be precisely the same; as may be seen fig. 2, plate 3, where the extrados is horizontal.

This construction is evidently the converse of that in Prop. 6, Sect. 1; and the demonstration of both depends upon the lemma to Prop. 5. Sect. 1; for in both it is supposed that the arch consists of an assemblage of beams kept in equilibrio by weights placed at the angular points.

Now, in the triangle $C p r$, from the proportion of the sides to the sines of the opposite angles,

we have $C p = \frac{s \angle C r p \times C r}{s. \angle C p r}$. Therefore

the weight represented by $C p$ will be proportional to this expression; and because $C r$ varies

as $\frac{1}{s. \angle r C p}$, if we substitute this last term in the

above value of $C p$, we shall have the weight or

force $C p$ proportional to $\frac{s. \angle C r p}{s. \angle C p r \times s. \angle r C p}$;

but $s. \angle C r p = s. \angle X C s$, or $s. \angle r C s$. There-

fore $C p$ is proportional to $\frac{s. \angle r C s}{s. \angle p C s \times s. \angle r C p}$.

We shall therefore have the following proportions for finding the weights or lines representing them; viz.

$$\frac{s. \angle t D t'}{s. \angle t D q \times s. \angle q D t'} : D q :: \frac{s. \angle r C s}{s. \angle r C p \times s. \angle p C s} : C p.$$

In numbers thus :

$$\text{Let } \angle C D E = 130^{\circ} \cdot 00'$$

$$\angle B C D = 165^{\circ} \cdot 00'$$

$$\angle A B C = 145^{\circ} \cdot 00'$$

And let the weight or force represented by $D q = 1.00$; then, because in the figure we have supposed $\angle C D E$ to be bisected by $D q$, we shall

have $\angle q D t' = 65^\circ$; and because $\angle p C s$ is the supplement of $\angle q D t$, or $\angle q D t'$, we have

$$\angle p C s = 115^\circ. 00'$$

$$\angle r C p = (165^\circ - 115^\circ) = 50^\circ. 00'$$

We may now determine the value of $C p$, $B o$, &c. in which the operation would be facilitated by using a table of logarithmic sines: but, to be more intelligible, we shall take the values of the trigonometrical terms from a table of natural sines, the labour of calculating such sines being too troublesome for practice. Thus,

$$\left. \begin{array}{l} \text{The natural sine of } \angle C D E \\ (= 130^\circ) \text{ or of } \angle r C p \\ (= 50^\circ), \text{ these angles being} \\ \text{supplements of each other} \end{array} \right\} = 7660444$$

$$\left. \begin{array}{l} \text{The nat. sin. } \angle q D t' \text{ or } \angle t D q \\ (= 65^\circ) \text{ or of its supplement} \\ \angle p C s (= 115^\circ) \quad . \quad . \quad . \end{array} \right\} = 9063078$$

$$\text{The nat. sin. } \angle B C D (= 165^\circ) = 2588190$$

Whence

$$\frac{7660444}{9063078 \times 9063078} : \frac{2588190}{7660444 \times 9063078} ::$$

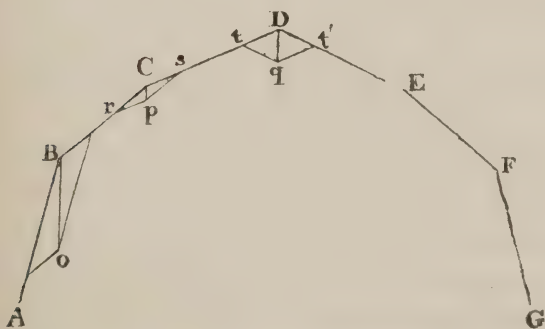
1.00 : .39972 nearly the value of $C p$.

In like manner $B o$ is found to be 3.1019 nearly.

PROPOSITION VI.

PROBLEM.

The weights or lines representing them, Dq , Cp , Bo , being given, as also the direction of the first line DC , or the angle Cdq , which it makes with the vertical Dq ; it is required to find the direction of the other sides, so that an equilibrium between the parts of the assemblage may take place.



Complete the parallelogram upon the diagonal DQ , make Cs equal to Dt , join sp , and make BC parallel thereto; then we shall have the

angle B C D, or the direction of the next side B C. Proceed in the same manner for the angle at B, and the direction of the side A B will be obtained.

Let the weights $D q = 1.00000$

$C p = 0.39972$

$B o = 3.10190$

Let the angle $q D t$, which the first line makes with the vertical, $= 65^{\circ} 00'$.

In the triangle $D t q$, by the proportion of the sides to the sines of the opposite angles, we have $s. \angle C D E$, or its supplement $D t q : D q :: s. \angle q D t : t q$, or its equal $D t$. Therefore

$$D t = \frac{D q \times s. \angle q D t}{s. \angle C D E}$$

$$\text{That is, } \frac{1 \times s. \angle 65^{\circ} 00'}{s. \angle 130^{\circ} 00'} = D t = C s = 1.1831.$$

Now the $\angle p C s$ or $p C D$ = the supplement of $\angle q D C = 115^{\circ} 00'$. The angle B C D is evidently composed of the angles B C p and p C D, and the angle B C p (Euc. Prop. 29, B. 1) = $\angle C p s$; we shall therefore have $\angle p C s$, and the sides C s and C p to find $\angle C p s$, to which $\angle B C p$ is equal.

And the angles B C p, p C D, are together equal to the angle B C D.

By plane trigonometry,

As, $C p + C s : C p - C s ::$ tangent of half the sum of the angles $C p s$, $C s p : \text{tangent of half the difference of those angles.}$

Or in numbers,

As, $1.1831 + .39972 : 1.1831 - .39972 :: \text{tangent } 32^{\circ} 30' : \text{tangent } 17^{\circ} 30'.$

Therefore $32^{\circ} 30' + 17^{\circ} 30' = \angle C p s = 50^{\circ} 00'.$

And $50^{\circ} 00' + 115^{\circ} 00' = \angle B C D = 165^{\circ} 00'.$

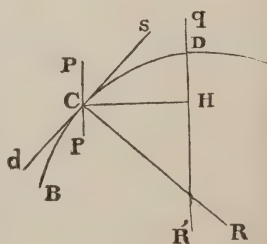
By a similar process we shall find the angle $A B C = 145^{\circ} 00'.$

If the other side of the figure $E F G$, be completed, and the fixed points be A and G , the opposite sides will mutually balance, and the equilibrium will be still preserved.

OBSERVATIONS.

The preceding propositions contain the principles of all that is necessary to be known by the practical builder for balancing the arch, be it of whatever form; it will be seen in the following pages that they are extremely fruitful in application, and equally useful in practice.

From the arrangement of lines forming the sides of a polygon, an equilibrium (as we have shewn) has been obtained. These lines may be diminished in length, and their number increased without end, in which case the polygon is exhausted or coalesces with a curvilinear arch, and it is evident that the result would be the same. It may not perhaps be unnecessary to observe, that in algebraical expressions of the equations of curves, and in fluxionary calculations relative thereto, the increments of the abscissæ of the curve are given, and ordinates determined with reference to the deflections of the sides of a polygon, which finally falls in with the curve. The sides of the polygon having been thus supposed indefinitely short, and the polygon exhausted, we shall have a curve $B C D$, and the angle $X C D$ (see fig. to prop. 5. sect. 1.) whose sine is equal to that of the angle $B C D$, will become the angle of contact $s C D$ in the figure, or that formed by a tangent to the curve at any point C , and the curve itself. Now the angle of contact de-



pend upon the degree of curvature of the arch at the point C, and diminishes as the radius of curvature increases, because then the curve approaches nearer to a coincidence with the tangent. Therefore we say the angle of contact is reciprocally proportional to the radius of curvature C R, directions for finding which, in all common curves, will follow in the next section. And the angles p C B, p C D, become equal to the angles p C d, p C s, which are supplements to each other; consequently their sines are equal, and equal to that of the angle p C s. Hence the arch being considered as a system of indefinitely short beams, the weight on an indefinitely small part of it at C, will be reciprocally as the radius of curvature at C, and the square of the sine of the angle made by the tangent to the curve and the vertical; that is as $\frac{1}{C R \times (\text{s. } \angle p C s)^2}$.

Now this would be sufficient for determining the conditions of equilibrium, if it were not that in an arch constructed to carry a load or part of an edifice, the weight must be diffused over every part of the arch, whereas this term expresses only the weight to be applied at the points of junction of the sides of the polygon,

which are considered as beams inflexible and without weight.

Suppose next, the curve of the arch to be divided into any number of equal parts, representing the breadths of the voussoirs or arch-stones in that direction, and to avoid complicating the subject by introducing pressures oblique to the curve, let the joints of the voussoirs all tend to the centre of curvature of the arch, or be perpendicular to the tangents at the places where the joints are situated. Then, from the nature of all curves, the horizontal breadths of the voussoirs increase from the springing part up to the vertex or crown. Now, every point on the surface of the voussoirs ought to be pressed by a weight proportional to the value of $C p$ just mentioned, but the vertical direction of the loading causes that loading to press on fewer points of the inclined voussoirs than of the horizontal one at the crown, and the points of pressure will be fewer as we approach the springing course. Therefore this deficiency of pressure on the inclined voussoirs must be compensated by a greater height of loading, and it is evident that the height, on this account, must be inversely proportional to

the breadth of the column over each voussoir; that is, to the horizontal distance between two verticals. But it is evident that these breadths are directly proportional to the sines of the angles between the tangents and verticals; that is to $s. \angle p C s$, for a column at C . It follows therefore that the heights of the verticals must, on this account also, be inversely proportional to $s. \angle p C s$: consequently the height $C p$, must, on all these accounts together, be proportional to

$$\frac{1}{C R \times (s. \angle p C s)^3}.$$

Or since $\angle p C s$ is equal to the complement of $\angle s C H$ which is the inclination of the curve at that point, to the horizon, we have $\frac{1}{s. \angle p C s} =$

$$\frac{1}{\cos. \angle s C H}, \text{ which by trigonometry, is equal to } \sec. \angle s C H.$$

Therefore we have $C p$ reciprocally proportional to the radius of curvature, and directly proportional to the cube of the secant of the curve's inclination, at that point to the horizon. But at D the inclination being nothing, its secant is equal to radius or unity: consequently the proportion for finding $C p$ becomes

$\frac{1}{D R'} : q D :: \frac{(\sec. \angle s C H)^3}{R C} : C p$; where
 $D R'$ and $R C$ are radii of curvature to the
 points D and C .

The above is a general expression applicable to all the curves that are capable of being applied in practice.

In order to keep in view the principle of the catenary; let it be observed that a thread or chain may be supposed attached at both ends to fixed points, and that from every part of the curve it assumes, chains may be attached, having their lengths so adjusted as, by their weight, to drag the original chain into any form whatever, suppose a semi-circle, semi-ellipse, or any other curve, forming what may be called a complex catenary. Then if an arch of the same form were erected in a vertical plane, and had a loading above it equal in height, over every part of the curve, to the lengths of the chains attached to the principal one; such an arch with its loading would stand and be in a state of equilibration.

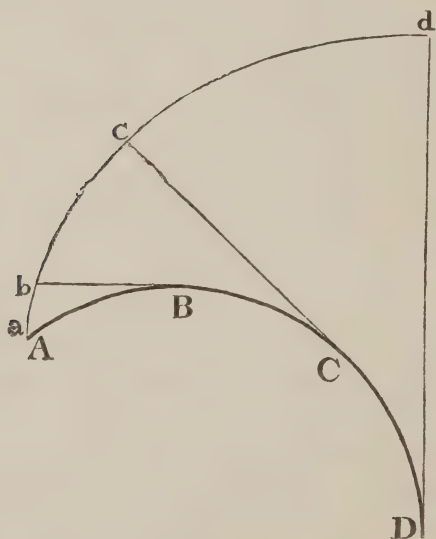
Now the formula just investigated shows how to determine the height of such loading over every part of an arch in a general way. Its application to particular curves will follow in the third section.

As the radii of curvature enter into the general expression of the height of the loading, it will be necessary to shew how these may be determined for the given curves before we proceed further.

SECTION II.

DEFINITION.

Of the Osculatory Circle, and the Method of finding the Radius of Curvature.



Let A B C D be a mould of wood, or other hard material, and let a thread be fixed to it at D, and kept close to its convexity, along the whole extent A B C D. Now taking the

thread by its end at A, and keeping it tort, let it be gradually unlapped or evolved from the mould. Then will its extremity A, describe another curve $a b c d$, called the involute curve of $A B C D$, or its evolatrix; $A B C D$ being the evolute curve.

It will be plainly perceived, that the involute curve has a momentary centre at every point in $A B C$, from which it is unwound; for instance, when it has been unlapped as far as B, and A has arrived at the same time at b , then B is a centre to b ; and if with the radius $b B$, there be described a circle touching the curve $a b$ in b , it will be the osculatory circle, or circle of curvature in that point, and $b B$ will be the radius of curvature.

All curves, a circle excepted, whose curvatures are neither infinitely great nor infinitely small, may be thus described by a thread evolved from a proper curve.

It is inferred that the unwrapped part of the thread is always a tangent to the curve $A B C$, and that it is perpendicular to the tangent of curvature at b .

In a circle the curvature is every where the same, so that the radius of curvature is constant; and the curvatures of different unequal circles

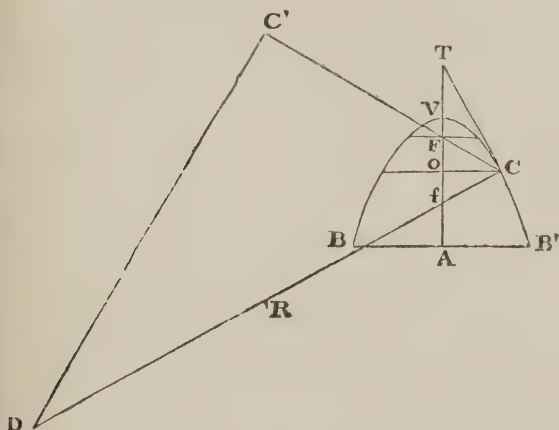
are reciprocally proportional to the diameters; that is, if one diameter be half the length of another, the curvature or convexity of a circle, having the former diameter, will be twice as great as that of the latter.

The demonstrations of the following propositions may be found in Vince's Fluxions and Conic Sections.

PROPOSITION I.

PROBLEM.

In any parabola B V B'. To find the diameter and radius of curvature to any point C, the ordinate O C to such point, being given, as also any other ordinate A B, and its corresponding abscissa V A.



Geometrically.

Draw the tangent C T to the point C, and find the focus F*. Through F produce C F in-

* To draw the tangent CT , let OC be drawn perpendicularly to the axis VA , and produce VA above the vertex indefinitely; make VT equal to OV , and join CT , and it will be a tangent to the parabola at the point C .

If Cf be drawn perpendicular to CT , cutting the axis

definitely, in which take $C C'$, equal to four times $C' F$; at C' and C , draw the lines $C D$, $C' D$, at right angles to $T C$ and $C' C$ respectively, intersecting each other in D . Bisect $D C$ in R ; then will $D C$ be the diameter, and $R C$ the radius of curvature to the point C .

Arithmetically.

Let the ordinate $A B' = 7.100$.

Its corresponding abscissa $V A = 10.000$.

Ordinate $O C$ to the point $C = 4.450$.

Then $A B'^2 : V A :: O C^2 : V O = 3.928$.

And $V A : A B' :: A B' : P$ (the parameter)
 $= 5.041$.

$$\frac{P}{4} = V F = 1.26025.$$

$$V O + V F = F C = 5.18855.$$

$$\frac{{}^*4 (F C^{\frac{3}{2}})}{\sqrt{V F}} = D C = 42.1114, \text{ the diameter of}$$

curvature at C :

$$\text{and } \frac{42.1114}{2} = 21.0557, \text{ the radius of curvature at } C.$$

in f , and $f T$ be bisected in F , then F is the focus of the parabola.

$\frac{{}^*4 (F C)^{\frac{3}{2}}}{\sqrt{V F}}$ signifies four times the square root of the third power or cube of $F C$, divided by the square root of $V F$.

At the vertex, the radius of curvature is equal to half the parameter.

PROPOSITION II.

PROBLEM.

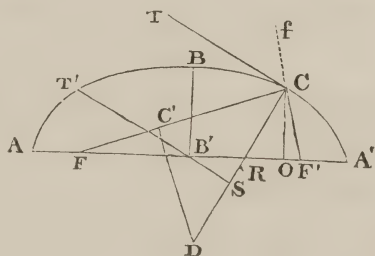
In any ellipsis A B A', to find the diameter and radius of curvature to any point C, the transverse and conjugate diameters being given, as also an abscissa A O to the ordinate C O at the point to which the radius of curvature is to be found.

Geometrically.

Draw the tangent C T*, and parallel thereto, through the centre B', draw T'S;

join C F, and in it produced, if necessary, make

C C' equal to $\frac{2 (T' B'^2)^2}{A B'}$. At



* From the point B as a centre, with A B' as a radius, describe an arc cutting A A' in F and F', which are the foci; and if lines be drawn from the foci to any point C, and one of them be produced, as F' C to f, and the angle F C f be bisected, a line drawn from C through the point of bisection will be the direction of a tangent to that point.

right angles to $C C'$ at C' , and to $T C$ at C , draw $C' D$, $C D$ intersecting each other in D ; $C D$ is the diameter of curvature required. Let $C D$ be bisected in R , then $R C$ is the radius of curvature.

NOTE.—In this and the following figure, $C' C$ is taken equal to one half only of the length required, on account of the inconvenient space these figures would otherwise have covered.

Arithmetically.

Let the transverse diameter $A A' = 20.00$,

the semiconjugate $B B' = 5.80$,

Abscissa (to $C O$) $A O = 16.00$,

Consequently $B' O = 6.00$.

Then by the properties of the ellipse :

$A' A^2 - (2 B' B)^2 = F' F^2$, and $F' F = 16.2924$,
nearly, which will be the distance between the foci :

$$\text{And } \frac{F' F}{2} + B' O = F O = 14.1462.$$

I. To find $C O$.

$$\begin{array}{l} A B'^2 : B' B^2 :: A O \times O A' : C O^2 = 21.5296 \\ \text{therefore } C O = 4.64 \end{array} \quad \left. \vphantom{\begin{array}{l} A B'^2 : B' B^2 :: A O \times O A' : C O^2 = 21.5296 \\ \text{therefore } C O = 4.64 \end{array}} \right\}$$

II. To find $C F$.

$$\begin{array}{l} F O^2 + C O^2 = C F^2 = 221.6445 \\ \text{therefore } C F = 14.8877 \end{array} \quad \left. \vphantom{\begin{array}{l} F O^2 + C O^2 = C F^2 = 221.6445 \\ \text{therefore } C F = 14.8877 \end{array}} \right\}$$

III. To find $C F'$.

$$A A' - F C = C F' = 5.1123$$

IV. To find $S C$.

$$* \sqrt{\frac{A B'^2 \times B' B^2}{C F' \times F C}} = S C = 6.6492$$

V. To find $T' B'$.

$$\sqrt{F C \times F' C} = T' B' = 8.723$$

VI. To find $D C$.

$$\frac{2 (B' T')^2}{C S} = D C = 22.88$$

the diameter of curvature to the point C .

$$\frac{22.88}{2} = R C = 11.44$$

the radius of curvature to the point C .

When the diameter of curvature to the vertex is sought :

$$\text{Then } \frac{2 (A B')^2}{B B'} = D C = 34.482$$

$$\text{And } \frac{34.482}{2} = R C = 17.241.$$

* $\sqrt{\frac{A B'^2 \times B' B^2}{C F' \times F C}}$ signifies that the product of the squares of $A B'$ and $B' B$, is to be divided by the product of $C F'$ and $F C$, and the square root of the quotient to be taken for $S C$.

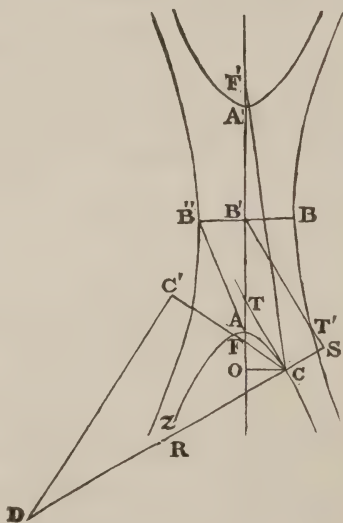
PROPOSITION III.

PROBLEM.

In any hyperbola $z A C$, to find the diameter and radius of curvature to any point C , the transverse and conjugate axes $A A'$ and $B'' B$ being given, as also an abscissa $A O$ to the ordinate $O C$, at the point to which the radius of curvature is to be found.

Geometrically.

Draw the tangent $C T^*$. Parallel thereto, and through the centre B' draw $B' T' S$; join $C F$, and produce it indefinitely towards C' . Make $C C'$ equal to $\frac{2 (B' T'^2)}{A B'}$, and at right angles to $C C'$ at C' , and $T C$ at C , draw $C' D$, $C D$ intersecting each other in D . $C D$ is the



* Join $B'' A$ and make $B' F$, $B' F'$ each equal thereto, then F and F' will be the foci of hyperbola.

diameter of curvature required; and if $C D$ be bisected in R , then $R C$ is the radius of curvature.

Arithmetically.

Let the transverse axis $A A' = 15.000$,

the semiconjugate $B B' = 3.104$,

Abscissa (to $C O$) $A O = 5.000$,

Consequently $B' O = 12.500$.

Then by the properties of the hyperbola,
 $B' A^2 + B' B^2 = B' F^2$. Whence $B' F = 8.1161$
 nearly, which will be the distance of either focus
 from the centre:

$$B' O - B' F = F O = 4.3839.$$

I. To find $C O$.

$$\begin{aligned} A B'^2 : B' B^2 :: A O \times O A' : C O^2 &= 17.1285 \\ \text{therefore } C O &= 4.1386 \end{aligned} \quad \left. \vphantom{\begin{aligned} A B'^2 : B' B^2 :: A O \times O A' : C O^2 &= 17.1285 \\ \text{therefore } C O &= 4.1386 \end{aligned}} \right\}$$

II. To find $C F$.

$$\begin{aligned} F O^2 + O C^2 &= C F^2 = 36.3477 \\ \text{therefore } C F &= 6.0288 \end{aligned} \quad \left. \vphantom{\begin{aligned} F O^2 + O C^2 &= C F^2 = 36.3477 \\ \text{therefore } C F &= 6.0288 \end{aligned}} \right\}$$

Join $F C$, $F' C$, and bisect the angle $F' C F$ for the direction of the tangent as in the ellipse.

III. To find C F'.

$$A A' + F C = C F' = 21.0288$$

IV. To find C S.

$$\sqrt{\frac{A B'^2 \times B' B^2}{F' C \times F C}} = C S = 2.067$$

V. To find B' T'.

$$\sqrt{F C \times F' C} = B' T' = 11.259$$

VI. To find D C.

$$\frac{2 B' T'^2}{C S} = D C = 122.67$$

the diameter of curvature at C.

$$\frac{122.67}{2} = R C = 61.33$$

the radius of curvature at C.

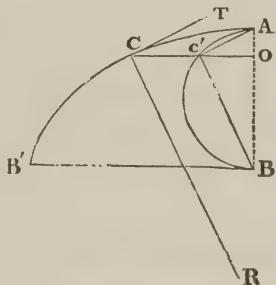
PROPOSITION IV.

PROBLEM.

In any cycloid B' C A, to find the diameter of curvature to any point C; the diameter of its generating circle A B, as also an abscissa A O to the point C, at which the diameter of curvature is to be found, being given.

Geometrically.

On the diameter A B, describe the generating circle B c' A, and join C O, cutting the same in c'; join c' A and c' B, and parallel to the former at C draw the tangent C T. At right angles to C T at C, draw C R, and make it equal to twice c' B. Then is R C the radius of curvature at the point C, and twice R C the diameter of curvature.



Arithmetically.

Let A B (diameter of generating circle) = 5.00

The abscissa A c' O = 1.00

Consequently O B = 4.00

Then, by the nature of the circle

$$\left. \begin{aligned} A O \times O B &= O c'^2 = 4.000 \\ \text{therefore } O c' &= 2.00 \end{aligned} \right\}$$

$$\text{And } O B^2 + O c'^2 = B c'^2 = 20.00$$

$$\text{therefore } B c' = 4.472$$

$$\text{And } 2 B c' = 8.944$$

the radius of curvature to the point C, for C R is equal to twice B c' by the nature of the cycloid; and

$$4 B c' = 17.888$$

the diameter of curvature.

PROPOSITION V.

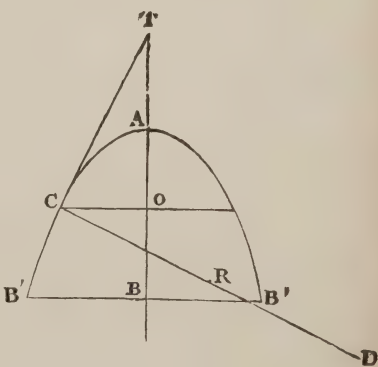
PROBLEM.

In any catenarian curve B' A B' to draw a tangent to any point C, the span B' B', and height A B of the curve, as also an abscissa A O to the ordinate O C, at which the tangent is to be drawn being given; and to the same point C, to find the radius and diameter of curvature.

Produce O A towards T, and in it take O T equal to $\frac{C O \times \sqrt{2 X \times A O + A O^2}}{X}$, where

X is put to represent the tension of the curve at the vertex*. Join T C and it will be the tangent required, and will $= O T^2 + O C^2$.

At C draw D C, B' at right angles to C T, and in it take

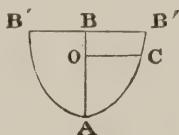


* It will be necessary to notice, that in the catenary there

C R, a third proportional to $X : \overline{X + A O}$, that is,

As $X : \overline{X + A O} :: \overline{X + A O} : C R$ (the radius of curvature to C) = 175.37.

is a peculiar property denominated its tension, that is, the degree of stretching it exerts at any point; and the ordinate to any of its abscissæ cannot be found without involving this tension or stretching in the calculation.



The tension at the vertex A may be found by the summation of the following series, calling it X.

$$X = \frac{1}{2} AB \times \left(\frac{B'B^2}{AB^2} + \frac{1}{3} - \frac{8AB^2}{45B'B^2} + \frac{691AB^4}{3780B'B^4} - \frac{33851AB^6}{453600B'B^6} \right),$$

&c., which is a series that will serve for every case, taking care to change the real values of AB and B'B, according to the height and semi-span given. Thus,

$$\text{Let } AB = 40$$

$$B'B = 50$$

Then it will stand thus :

$$X = \frac{40}{2} \times \left(\frac{25}{16} + \frac{1}{3} - \frac{128}{1125} + \frac{176896}{2362500} - \frac{97693696}{7087500000} \right), \text{ \&c.}$$

Put these fractions into decimals, to sum them up more easily, and then

$$X = 20 \times (1.5625 + .3333 - .1137 + .0749 - .0138, \text{ \&c.} =$$

$20 \times 1.8432 = 36.864$; but if the series had been carried a term further (which is not necessary for practical purposes) it would have come out 36.88, or 36.9 nearly, as Dr. Hutton makes it in his Principles of Bridges (page 37, second edition, London, 1801).

Any ordinate OC to a given abscissa, may be found by a

Hence $2 \times 175.37 = 350.74$ (the diameter of curvature DC).

In the above the height $AB = 50$, $B'B = 30$, and $AO = 25$.

summation of the following series. Let it be to an abscissa $AO = 40.00$. The constant tension ($= X$) being as above 36.9 nearly. Then

$$OC = \sqrt{2 \times X \times AO} \times \left(1 - \frac{AO}{12X} + \frac{3AO^2}{160X^2} - \frac{15AO^3}{2688X^3} + \frac{105AO^4}{55296X^4} \right), \text{ \&c.}$$

Putting these into numbers as before, we shall find

$$OC = 49.835 \text{ or nearly } 50.00.$$

SECTION III.

On the comparative strength of arches, and the method of finding the extrados of an arch from a given intrados.

From what has been observed at page 25, it may be inferred, that the strength of one part of an arch to that of another, will be proportional to the greatest weights those parts are capable of bearing; that is, as the cube of the secant of the curve's inclination to the horizon at those points, divided by the radii of curvature; and when two arches formed of similar materials have the same span and height, their comparative strengths at any two corresponding points will be also in the same proportion; but, since if one part of an arch fails, the whole will fall to ruin, and as the crown is the weakest part in all arches, it will only be necessary to compare them with each other at that point. Now as all curves at their vertices have no inclination to the horizon, the angles thereat will always be $0^{\circ} : 0^{\circ}$, and the secants become the

radii of those angles, and are the same or equal in every curve; whence it follows, that the comparative strength at the crowns, of two arches having the same span and height, is reciprocally as the radii of curvature at those points.

Taking for example an arch 100 feet span and 40 rise; the radii of curvature at the crowns in the different curves will be as follow :

	FEET.			
Segment of a circle	.	.	.	51·25
Parabola	.	.	.	30·125
Ellipsis	.	.	.	62·5
Hyperbola	.	.	.	37·417
Catenary	.	.	.	36·9

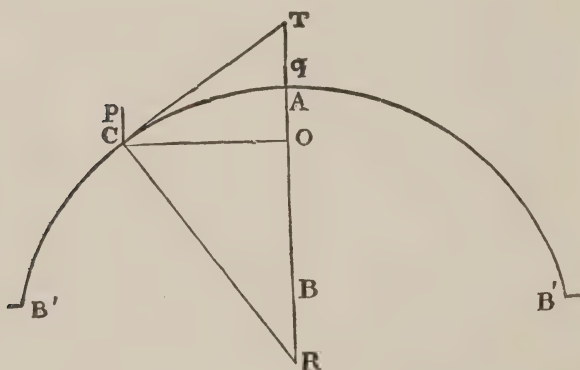
Hence, *cæteris paribus*, at the crowns of a parabola and an ellipse, the strength of the former to the latter will be as 62·5 to 30·125, &c., &c.

This will be more easily perceived if we consider the arches as polygonal, the sides being infinitely small. When the two first sides form a very large angle with each other, so that their mutual inclination approaches nearly to a straight line, the curvature is very small, and

they will evidently have less power to resist a pressure on the angle at the vertex, than when inclined to each other in a smaller angle, wherein the curvature is increased, and the radius of curvature consequently diminished. There are few, if any, instances of elliptical arches of large span in proportion to their height, in which the crowns have not sunk considerably.

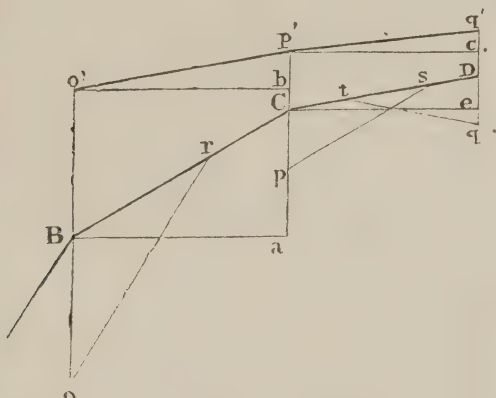
PROPOSITION I.

To find the height of the superincumbent wall, or extrados, above every point of a circular arch, so that by its pressure all the parts of the arch may be kept in equilibrio.



Let $B' A B'$ be the segment of a circle, whose span is $B' B'$, and height $A B$; R the centre, and $A R$ the radius. Also let $A q$ be the height of the wall at the vertex, and $A O$ an abscissa to the ordinate $O C$ at the point C , above which the height of the wall is to be found.

If it be required to find, arithmetically, the heights of the several lines p' C, o' B, &c. the following is an easy process.



$$\begin{array}{rcl} \text{Let} & q' D & = 1.00 \\ & o' p' & = 4.90 \\ & p' q' & = 4.25 \\ \angle p' q' D & = & 85^{\circ} 00' \\ \angle o' p' C & = & 80^{\circ} 00 \\ \angle C D q & = & 80^{\circ} 00 \end{array}$$

Having let fall the verticals $q'D$, $p'C$, $o'B$, draw $p'c$, Ce , and $o'b$, at right angles thereto.

Then by plane trigonometry,

$$p'c = \frac{p'q' \times s. \angle p'q'c}{s. \angle p'cq'} = 4.233$$

$$e D = \frac{C e (=p' c) \times s. \angle D C e}{s. \angle C D q} = 0.7465$$

$$c q' = \frac{p' c \times s. \angle q' p' c}{s. \angle p' q' c} = 0.3703$$

$$c e = p' C = p C = D q' + D e - q' c = 1.0 + 0.7465 - 0.3703 = 1.3762.$$

Now

$$D t = \frac{D q \times s. \angle C D q}{s. \text{ of } (180^\circ - 2 \angle C D q)} = 2.879 (=C s).$$

Also

$C p + C s : C s - C p :: \text{tangent } \frac{1}{2} \text{ sum of the angles } C p s, C s p : \text{tangent } \frac{1}{2} \text{ difference of those angles.}$

So that $\angle C p s = 56^\circ 31'$

$\angle C s p = 23^\circ 29'$

$$o' b = \frac{o' p' \times s. \angle o' p' b}{s. \angle o' b p'} = 4.823 (=B a)$$

$$p' b = \frac{o' p' \times s. \angle p' o' b}{s. \angle o' b p'} = 0.7941.$$

As the angle $B C a$ is equal to the angle $C p s$, draw $B a$ parallel to $o' b$, then $B a C$ is a right angle; consequently $\angle a B C = 90^\circ - 56^\circ 31' = 33^\circ 29'$,

And

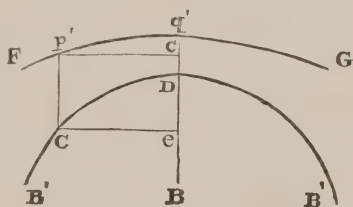
$$C a = \frac{B a (= o' b) \times s. \angle a B C}{s. \angle B C a} = 3.192$$

$$o' B (= b a) = C p' + C a - b p' = 1.376 + 3.192 - .7941 = 3.774.$$

The whole extent is $B a + C e (= p' c) = 4.823 + 4.233 = 9.056$, and height $= C a + e D = 3.192 + 0.7465 = 3.9385$.

Many intradoses may be found by the above method to a single given extrados, the span of each depending upon the degree of inclination given to the first line $D C$; that is, if the angle $q' D C$ were only $92^{\circ} 00'$, the intrados will be of less height or rise in proportion to any given span, than if the said angle were $95^{\circ} 00'$. Thus the architect has the means of choosing the most convenient and pleasing form for his intrados as circumstances may require*.

* If, having an extrados of a given curve, it be required to adapt an intrados thereto, whose span and height must be of certain given dimensions, the following operation will be necessary ;



Let $F p' q' G$ be an extrados, as a segment of a circle, ellipse, &c. &c. whose ordinate $p' c$, and abscissa $q' c$, are known at every point thereof.

If, instead of treating the arch as an assemblage of beams with weights at the angles, we

Let q' D (height at crown)	=	7.00
q' c the abscissa	=	3.00
C e, and p' c its common ordinates (supposing, in the present example, the extrados to be the segment of a circle whose radius is 230) will	=	34.00
B B' the half span	=	50.00
D B the height or rise	=	40.00

To find D e.

Put C e (=p' c) =	y
B B'	s
D B	h
D e	x

$$\text{I. } p' C = e c = C \times \frac{\dot{y} \ddot{x} - \dot{x} \ddot{y}}{\dot{y}^3} = \frac{C}{\dot{y}} \times \text{fluxion of } \frac{\dot{x}}{\dot{y}}.$$

$$\text{It is evident } e c = q' D + D e - q' c = \frac{C}{\dot{y}} \times \text{fluxion of } \frac{\dot{x}}{\dot{y}}.$$

$$\text{II. Assuming } \dot{y} = \frac{\dot{x}}{z} \text{ we have } z = \frac{\dot{x}}{\dot{y}} \text{ and } \frac{C}{\dot{y}} \times \text{fluxion of}$$

$$\frac{\dot{x}}{\dot{y}} = \frac{C z \dot{z}}{\dot{x}}$$

$$\text{That is } 7 + x - 3 = \frac{C z \dot{z}}{\dot{x}},$$

$$\text{And } x + 4 = \frac{C z \dot{z}}{\dot{x}}, \text{ or } x \dot{x} + 4 \dot{x} = C z \dot{z}.$$

$$\text{The fluent of this expression is } \frac{x^2}{2} + 4x = \frac{C z^2}{2},$$

were to consider it as covered by a loading diffused over it, it is evident, from the observations

$$\text{or } 2x^3 + 16x = 2Cz^2, \text{ or } x^2 + 8x = Cz^2;$$

$$\text{from which } z = \sqrt{\frac{x^2 + 8x}{C}}.$$

III. Now as $\dot{y} = \frac{\dot{x}}{z}$,

$$\dot{y} = \dot{x} \div \sqrt{\frac{x^2 + 8x}{C}} = \dot{x} \sqrt{\frac{C}{x^2 + 8x}} = \dot{x} \sqrt{\frac{C}{x^2 + 2 \times 4x}}.$$

The fluents of which are $y = \sqrt{C} \times$ hyperbolic logarithm
of $(x + 4 + \sqrt{x^2 + 8x})$

but at the vertex where $x = 0$ we have

$$y = \sqrt{C} \times \text{hyperbolic logarithm of } 4;$$

So that the corrected fluent becomes $y = \sqrt{C} \times \text{hyp. log.}$

$$\text{of } \frac{x + 4 + \sqrt{x^2 + 8x}}{4}.$$

IV. To obtain the constant quantity C . When e arrives at B ,
we have $x = DB$ which call h ,

$$y = B'B \text{ which call } s,$$

and the abscissa of the extrados = 5.5.

Then repeating the above operation, we obtain

$$s = \sqrt{C} \times \text{hyp. log. of } \frac{h + 1.5 + \sqrt{h^2 + 3h}}{1.5},$$

$$\text{Consequently } \sqrt{C} = s \div \text{hyp. log. of } \frac{h + 1.5 + \sqrt{h^2 + 3h}}{1.5}.$$

V. Hence we obtain,

* The hyperbolic logarithm of any number may be found by multiplying its common or Briggs's logarithm by 2.302585, &c. the product being the hyperbolic logarithm of the number required.

at the end of Section I. that a nearer approximation to the true form of the intrados would be

$$s \times \text{hyp. log. } \frac{x + 4 + \sqrt{x^2 + 8x}}{4}$$

$$y = \frac{\text{hyp. log. } \frac{h + 1.5 + \sqrt{h^2 + 3h}}{1.5}}{\text{hyp. log. } \frac{x + 4 + \sqrt{x^2 + 8x}}{4}}$$

$$\text{or } \frac{y}{s} = \frac{\text{hyp. log. } \frac{h + 1.5 + \sqrt{h^2 + 3h}}{1.5}}{\text{hyp. log. } \frac{x + 4 + \sqrt{x^2 + 8x}}{4}}$$

$$\text{Hence } \frac{y}{s} \times \text{h. l. } \frac{h + 1.5 + \sqrt{h^2 + 3h}}{1.5} = \text{h. l. } \frac{x + 4 + \sqrt{x^2 + 8x}}{4}.$$

$$\text{Suppose the last expression } \frac{y}{s} \times \text{h. l. } \frac{h + 1.5 + \sqrt{h^2 + 3h}}{1.5}$$

when put into numbers to be a hyperbolic logarithm, and find the natural number corresponding to it, which call N ;

$$\text{then } N = \frac{x + 4 + \sqrt{x^2 + 8x}}{4},$$

$$\text{or } 4 N = x + 4 + \sqrt{x^2 + 8x},$$

$$\text{and } 4 N - x - 4 = \sqrt{x^2 + 8x},$$

Squaring both sides, we have

$$16 N^2 - 8 N x - 32 N + x^2 + 8 x + 16 = x^2 + 8 x,$$

$$\text{or } 16 N^2 - 32 N + 16 = 8 N x ;$$

$$\text{whence } x = \frac{2 N^2 - 4 N + 2}{N}.$$

It is only in the second and third steps that a change is necessary ; namely, where the value of the abscissa of the extrados varies according to the circumstances of the data. Any

obtained by multiplying the heights Dq , Cp' , Bo' , &c., found as above, by radius and by the secants of the angles $D Ce$, $C Ba$, &c., respectively.

person the least acquainted with common algebra may carry this operation through.

In numbers as follows :

$$\frac{y}{s} = \frac{34}{50} = .68, \text{ and as } h = 40, \text{ the expression } \frac{h + 1.5 + \sqrt{h^2 + 3h}}{1.5}$$

becomes $\frac{40 + 1.5 + \sqrt{1600 + 120}}{1.5} = 55.31$, the hyperbolic logarithm of which is 4.01297 , this being multiplied by $.68$ gives 2.72884 , and looking into a table of hyp. logs. we have the natural number corresponding to this $= 13.13$, which we call N ;

$$\text{then } x = \frac{2N^2 - 4N + 2}{N} = \frac{344.8 - 52.52 + 2}{13.13} = 22.4$$

for the length of the abscissa De , whence the value of ce , that is of $p'C$, is known.

For an horizontal extrados (see Hutton's Principles of Bridges, sect. 3.) the equation to the curve of the intrados is found to be

$$y = s \times \frac{\text{hyp. log. of } \frac{a + x + \sqrt{2ax + x^2}}{a}}{\text{hyp. log. of } \frac{a + h + \sqrt{2ah + h^2}}{a}} \text{ where } a \text{ ex-}$$

presses $q'D$, the height at the crown.

CONCLUDING OBSERVATIONS ON THE EQUILIBRIUM OF ARCHES.

It has long been a subject of complaint, that the theories of the statical equilibrium of bodies afford little assistance in the execution of any works where it is of importance to obtain a requisite degree of stability with the least possible expense of materials. Yet it can hardly with justice be considered as a reproach to mathematicians that they have not been able to discover theories which should comprehend all the relations of the subjects involved, and which should be applicable to the actual circumstances of every case, since this would require a more intimate knowledge of the laws of mechanical action than we at this time possess.

Under the disadvantages arising from our imperfect acquaintance with the properties of bodies, the only thing which can be done is to assume the absolute perfection of all the qualities of the materials employed, to investigate the conditions of equilibrium according to that supposition, and then, to leave to the persons who are to apply the conditions practically,

the care of making such modifications in them as are indicated by the results of the best experiments that have been tried.

The objections that are made to the theory of arches just delivered are, that the voussoirs are supposed indefinitely thin, and that the loading above is supposed to act only in the vertical direction. Now, with respect to the first, the smallest excess of pressure on any part of the arch, above what the theory assigns, would force the voussoirs from their places; and with respect to the second, when the loading consists of loose material, as rubble or gravel, it exerts pressures laterally as well as vertically.

The latter point may be dispatched in a few words; the principal part of that lateral pressure is counteracted by the resistance of the piers, and the remainder is too inconsiderable to deserve notice. It may also be observed that, of late, the practice of filling up the haunches with loose material seems to be abandoned, and that of solid walls, parallel to the length of the bridge, to be adopted. This work not only presses vertically upon the arch, but, by its cohesion, renders an excess of weight at the crown less likely to force up the arch at the haunches.

The weakness arising from the other circumstance would be of serious consequence, if it were not that the artists have invariably given to the voussoirs a certain degree of thickness, in order that the divergency of the joints may be sufficient to allow them to keep their places. This is certainly an indispensable requisite ; and as the tendencies of these stones downward upon their joints produce pressures, which are different from those produced by the vertical loading in the catenarian theory, it may be expected that attention should be paid to something more than the equilibrium produced by such vertical loading.

The simplest method of adapting the catenarian theory to practice, seems to be that of making the courses of voussoirs balance themselves, as in Mr. Atwood's theory, viz. by equalising the tendency which any given course has to rise, by a resultant of the lateral pressure of the superior courses, with the tendency downward in an opposite direction by a resultant of the weight of the given course. The weights of the voussoirs, and consequently their lengths being determined by such means, we may consider the extrados of these voussoirs as a new intrados, and proceed to determine a new extrados

for the roadway by the heights of the columns of vertical loading, according to the catenarian theory: and as the exterior curve of the voussoirs will always be more flat than the interior curve, and will always rise at an acute angle with the horizon, it is manifest that the final extrados will approximate nearer to a right line, and will consequently be more convenient for a roadway than that which would be obtained either by the catenarian theory in its simple state, or by the method of equilibrated voussoirs.

Let it also be recollected, that the voussoirs or stones of which the arch is composed, have flat surfaces in contact with each other, and that the cement and friction add to their stability. It is true that no theory will shew what additional weight (in practice) an arch can sustain on its weakest part; but an arch of equilibration is certainly better able to sustain a greater weight accidentally placed on any part than another arch would be.

Strict mathematical precision cannot always be obtained; but by attention to what has been shewn, the architect will be better able to combine the requisite strength with that beauty which it will ever remain in his province alone to impart to the design.

Of the thickness given to the voussoirs at the crown, there have been many differing examples; in general, we may put it down from $\frac{1}{14}$ to $\frac{1}{17}$ of the span of the arch, varying according to particular circumstances. Palladio gives the thickness of them in the ancient* bridge at Rimini, at $\frac{1}{10}$ of the span to the middle arch, and $\frac{1}{8}$ to the side arches.

Again in the bridge at Vicenza, he gives the voussoirs to the middle arch $\frac{1}{12}$ of the span, and the side arches $\frac{1}{9}$. But in a design by himself†, he makes the thickness of the voussoirs in the middle arch $\frac{1}{17}$, and that of those of the side arches $\frac{1}{14}$ of the span.

L. B. Alberti recommends‡ them to be of the largest and hardest stones; and directs no stone to be used that is not at least $\frac{1}{10}$ of the chord of the arch; nor (says Alberti) should the chord itself be longer than six times the thickness of the pier, nor shorter than four times.

In Westminster bridge the voussoirs are about $\frac{1}{15}$ of the span, and at Blackfriars bridge nearly the same.

* ——— il lor modeno e per la decima parte della luce de' maggiori e per l'ottava parte della luce de' minori. Cap. 11. lib. 3.

† Lib. 3. cap. 14.

‡ Lib. 4. cap. 6.

It is to be observed that in most of, if not in all, their bridges, the ancients did not increase the dimensions of their voussoirs from the crown towards the springing or coussinet, but made them of an equal thickness throughout; in this they were followed by Palladio, and all the Italian architects. They sometimes, however, made the alternate voussoirs larger than the others, as Mr. Labelye has done in Westminster bridge.

While the voussoirs are considered as indefinitely short, and are held in a state of tottering equilibrium by the vertical pressure of the superincumbent loading alone, as the theory, in its simple state requires, the arch would not be calculated to support any extraneous weight. In practice, the voussoirs are of considerable length, and their adjacent surfaces are in contact; and when an additional weight is brought to act upon any part, suppose at the crown, it will cause the joints at such part to open on the concave side; the haunches will consequently be forced up, and their joints will open on the convex sides: but while the imaginary lines, expressing the directions of pressure passing through the voussoirs, are not so much distorted as to be thrown out of the limits of the

surfaces of the blocks, the arch will stand, though loaded at the crown with a certain degree of weight beyond what the strictness of the theory allows; when these imaginary lines are removed out of the limits of the adjacent blocks or voussoirs, the arch will be completely destroyed. It therefore follows that the voussoirs should be as large as may conveniently be got; the larger they are the more may the distortion be increased, without endangering the structure, since the directions of their pressures will be less likely to exceed the limits of their magnitude.

In general, while a line can be drawn from the crown to the haunches, passing entirely within the surfaces of the voussoirs, the arch will stand; but, when any part of the line falls out of their surfaces, the stability of the arch is instantly destroyed.

Enough, it is hoped, has been said to convince the artist of the necessity of balancing his arch with caution, and as exactly as circumstances will allow; and where an equilibrium according to this theory cannot be obtained, the fertile mind of an ingenious artist will naturally furnish expedients to obviate the inconvenience likely to arise from a want of it.

SECTION V.

On the horizontal drift or shoot of an arch, and the thickness of the piers.

PRELIMINARY OBSERVATIONS.

If a bridge should consist of but one arch between the abutments, and each abutment does not immediately rest against an immovable object, such as the bank of a river, it will be evident, if the materials composing the arch have a tendency to yield to any pressure in the direction of the length of the bridge, that the effect of this lateral thrust will be either to push the piers off horizontally, or to overturn them. The former effect may take place if the curve of the arch should begin near the foundations; the latter, if it rest upon piers considerably elevated. In either case the thrust must be counteracted, by giving a proper thickness to the abutments.

If a bridge consist of several arches, and we choose to consider the lateral thrust of each arch alone, without regard to the contrary

thrusts of those with which it is connected, the counteraction must be effected by the same means.

No theory, purely mathematical, has yet been discovered for determining the equilibration of an arch in this respect, nor have all the circumstances attending this species of pressure been satisfactorily ascertained. The principle which has been generally assumed as the basis of the investigation is, that the materials of which each semi-arch as A C D B (see the figures to Plate 1) is composed, have a freedom of motion and a tendency to roll or slide over the intrados B D, by a resultant of the action of gravity, and that they are retained by the side B A of the wall or pier. The equivalent of all these forces being estimated in some particular direction, is considered as the force which tends to push off or overturn the pier or abutment.

To determine this force, it becomes necessary to find the centre of gravity of any longitudinal section of a semi-arch; for in that point, by the laws of mechanics, the mass of materials may be considered as collected. Or, if the force is to be estimated in a horizontal direction, it will be merely necessary to determine the situation of a vertical line passing through the centre of

gravity, which is more easy than to determine that of the centre of gravity itself.

If the arch be perfectly equilibrated by its loading, according to the rules delivered in this treatise, and if only such a portion of the arch is employed as will permit its extrados to serve accurately for the intended roadway, then the two following propositions will shew how, in a very easy manner, to determine the situation of the vertical passing through the centre of gravity, and how from thence to determine the lateral thrust exerted by the arch.

For the arch line B' D with its loading may be considered as resting on the points B' and D,

and exerting a pressure there in the directions of the tangents $t B'$, $t D$; the re-actions of the materials which support the half-arch at those points are consequently to be considered as forces in the opposite directions $B' t$, $D t$. But in mechanics, where a body is sustained by two forces, those forces produced will meet, either in the centre of gravity of the body supported, or the centre of gravity will be in a vertical line passing through that point of intersection.

EXAMPLE.

$$\text{If } \angle f B' t = 84^\circ 46'$$

$$\text{and } \angle B' f t = 90^\circ 00'$$

$$\angle B' t f = 5^\circ 14'$$

$$\text{Then } t f = B D \text{ being } = 40^\circ 00'$$

$$\text{and } B B = 50^\circ 00'$$

By plane trigonometry,

We shall have

$$B' f = \frac{D B \times s. \angle B' t f}{s. \angle f B' t} = 3.699$$

$$\text{and } f B = 40 - 3.699 = 36.301.$$

When the curve thus loaded to equilibrium is one of those mentioned in note to Prop. 1,

Find the area of $B' F q D$ which may represent the weight of the semi-arch, since the weight is proportional to the area of the section. Then by mechanics as

$t f : B' f ::$ weight of the semi-arch : its
shoot or horizontal drift in the direc-
tion $f B'$.

OBSERVATIONS.

The method of estimating the lateral thrust, contained in the two preceding propositions, must not be employed when the arch rises perpendicularly, or nearly so to the horizon. In fact, the intersection t of the tangents $D t$, $B' t$ would then take place upon, or very near to the vertical line $B' F$, and the semi-arch would appear to have no lateral thrust, a circumstance which, though true in theory, must not be admitted in practice. For as the extrados cannot, on account of its great height over the springing courses, serve for a road, it is evident that by cutting off the part above the line of the intended road, the position of the centre of gravity is altered, and to determine the vertical line $t f$, the centre of gravity of the semi-arch must be found by one of the usual methods. The easiest

would undoubtedly be to cut the form of the half arch in pasteboard, and make it support itself upon a point, the point thus found will be the place of the centre of gravity, which suppose to be at G: then the horizontal shoot or drift may be found as in Prop. 2, of this Section.

Now to determine the thickness of a pier necessary to counteract the horizontal thrust of the arch, we must consider that this thrust is to be resisted by the friction which the stones composing the pier experience in sliding upon each other. From sundry experiments it has been found, that in some kind of stone the friction of one block moving horizontally upon another, is equal to one third of the weight of the moving block. If we adopt this determination, the weight of the pier ought to be equal to three times the horizontal drift of the arch to produce an equilibrium.

If A = the area of the semi-arch, which area may represent its weight: then, by Prop. 2, we have $-\frac{B' f \times A}{t f}$ = the horizontal thrust; hence $\frac{3 B' f \times A}{t f}$ = the area of the vertical section FR of the pier, which area may represent the weight

of the pier, and $\frac{3 B' f \times A}{t f \times X Y}$ = the thickness R Q required.

The specific gravity of the materials composing the arch and pier, is here supposed to be the same, which is generally the case, and on this account, it does not enter into the formula. The height X Y is given, and we suppose the whole pier to stand dry, or out of the water.

If we could determine the conditions of equilibrium between the stability of the pier and the effort of the arch to overturn it about R, we might estimate the horizontal thrust as before; thus let G be the centre of gravity of the half arch; draw G C parallel to B' B, and suppose the horizontal thrust to be applied at C, at the extremity of the arm C Q of the bent lever C Q R. In this case $\frac{B' f \times A \times C Q}{t f}$ will express the effort of the arch to overturn the pier. But F Q $\times$ R Q is equal to the area of the section of the pier, and represents its weight; and this weight is supposed to act in the vertical line X Y, passing through the centre of gravity of the pier, it consequently acts at the extremity of the lever R Y, which is equal to $\frac{1}{2}$ R Q.

Therefore we have $F Q \times \frac{1}{2} R Q^2$ to express the stability of the pier. Then equating these two forces and reducing, we have

$$R Q = \sqrt{\frac{2 B' f \times A \times C Q}{t f \times F Q}} \text{ for the}$$

thickness required, supposing as before, the whole pier to stand out of water.

But inasmuch as part of the pier is in most cases immersed in water, it thereby loses so much of its weight as is equal to the weight of a quantity of water, whose bulk is equal to that of the immersed part of the pier. To allow for this, let us suppose the material of the bridge to be granite, the specific gravity of which, to that of water is as $3\frac{1}{2}$ to 1, and let the pier stand in water as high as the springing B' . Then the stability of the pier will be expressed by $1.75 F Q \times R Q^2 - 0.5 B' Q \times R Q^2$, and the effort of the arch to overturn it will be $\frac{3.5 B' f \times C Q \times A}{t f}$; equating these terms,

$$\text{we have } R Q = \sqrt{\frac{7 B' f \times C Q \times A}{t f (3.5 F Q - B' Q.)}}$$

The thickness of the piers determined on this principle will be found rather greater than is allowed by modern architects, and to reduce the results of theory nearer to the general practice,

mathematicians have brought the point of application of the thrust of the arch nearer to the foot than the point C is; but as this is rather arbitrary, it has been thought better to leave it in a horizontal line passing through G. It is for the architect to exercise his judgment in making such modifications of the theory.

It is advisable (if possible) to construct the pier of heavier materials than the arch, as by that means it will not require so great a thickness, and a greater waterway* will consequently be obtained.

If the pier be constructed as we have supposed above, that is, independently of the resistance arising from the abutments, or from the neighbouring arches, a considerable saving of expense in the centering of a bridge, where several arches are required, will accrue; for allowing the piers to be individually capable of

* In the Philosophical Transactions 1758, is a paper by Robertson, on the fall of water under bridges, wherein the following formula is given.

$$\left(\left(\frac{25}{31} \frac{b}{c} \right)^2 - 1 \right) \cdot \frac{v}{4a} = \text{fall of the river occasioned by the piers.}$$

Where b = the breadth of the river,

c = the waterway,

v = the velocity of the river in one second,

a = the fall of a heavy body in one second =

16·0899.

resisting the drift, the centre of one arch may be struck before another is begun, and it will serve for its opposite and corresponding one, that is, supposing them to be of equal sizes, which is, unless under particular circumstances, always the case. Otherwise the arches depend, for their stability, upon the counteracting efforts of each other, and additional centering, one of the most expensive pieces of machinery used in their construction, must be necessarily employed.

The Italian as well as the ancient architects seem to have regulated the thickness given to the piers of bridges, by proportioning them to the span of the arches, without regard to any other circumstances.

In the little bridge over the Ilyssus, near Athens, the middle arch of which is only 19 feet 10 inches span, the extraordinary thickness of nearly 8 feet 6 inches is given to the piers, being nearly five-twelfths of the span.

In the bridge at Rimini, admired by Palladio for its beauty* as well as strength, nearly the same proportion is observed†.

* “ Mi pare il piu bello e il piu degno di considerazione, si per la fortezza, come per il suo compartimento.”

† “ I pilastri sono grossi poco meno della mettà della luce degli archi maggiori.” Lib. 3, cap. 11.

In the bridge over the Bacchilione, at Vicenza, the piers are one-sixth of the span of the middle arch; and in a design by Palladio himself, he makes the pier to the middle arch one-fifth of the span.

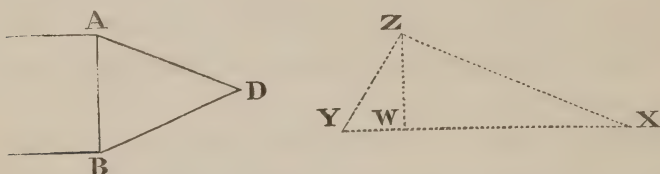
L. B. Alberti says, that the piers ought not to be less than one-sixth, nor more than one-fourth of the span.

At Westminster bridge the piers to the middle arch are a little more than one-fourth of the span, and at Blackfriars bridge a little more than one-fifth.

It is evident, however, after what has been shewn, that in order to estimate the proper thickness, the circumstances of the span, height, forms of the extrados and intrados, must enter into the consideration.

The extremities of the piers of bridges are usually terminated in points towards the current of a river, in order to diminish the pressure of the water against the piers, and lessen the eddy which the water, flowing off laterally after striking the piers, forms at the shoulders. Now, to compare the pressure sustained by a pier terminating in a head, as A B, at right angles with the stream, with that sustained by one whose extremity

forms an isosceles triangle as $A D B$ on the plan :
let $X Y$



parallel to the length of the pier represent the force with which the water would strike any point of the square head $A B$: this force may be resolved into $X Z$, which is parallel and $Z Y$ which is perpendicular to the face $A D$. The portion of the force of water represented by $X Z$ being parallel to $A D$ produces no effect upon it, and the remaining force $Z Y$ may be resolved into the two $Z W$ perpendicular to, and $W Y$ coincident with the length of the pier: the portion of the force of the water represented by $Z W$ is counteracted by an equal force in an opposite direction, and there only remains the force represented by $Y W$ tending to push the pier from its place, consequently the pressure of the current against a pier terminated perpendicularly to the stream, as $A B$ is to the pressure against one terminated by the triangle $A D B$, as $X Y$ to $W Y$.

If the angle at D were a right angle, the force against A D B would be only half the force against A B, and it is evident that the force must become less in proportion as the angle becomes more acute; and in fact it may be shewn that the force is inversely proportional to the square of the side A D.

It is evident too, that curvilinear piers oppose less resistance to the stream the nearer their faces approach to right lines; but care should be taken not to make the angle at D too acute, on account of the injury which vessels may sustain by being driven against it, and the eddy which is produced when the current sets obliquely to the pier.

Though some objections have been made to this theory, nothing better has yet been submitted to calculation.

OF DOME VAULTING.

A work of this kind would be incomplete if it did not contain some account of the principles of equilibrium in domes of masonry; it is therefore purposed to conclude this section by a few words on that species of building.

Let Figure 1, Plate 4, represent part of a vertical section through the centre of the dome; let AB represent one half of the keystone, and the line AB the direction in which it presses against the adjacent stone BC , that is perpendicular to the joint B : find g the centre of gravity of the half keystone; draw the vertical line gv , and make it of any length at pleasure to represent the weight of the half-stone AB , and through v draw vf at right angles to gv , meeting AB produced in f . If gv represent the weight of the stone AB , then will gf represent its pressure against the adjacent stone BC , in a direction perpendicular to the joint B . Now make Bb equal to and in the same direction as gf , and on an indefinite vertical line drawn through B take Bc to represent the weight of the stone BC , complete the parallelogram cb , and draw the diagonal Bd ; then will Bd represent the quantity and direction of the pressure of the two stones AB , BC upon the joint C . Again, make Ce equal to and in the same direction as Bd , and on an indefinite vertical line through C take Ch to represent the weight of the stone CD ; complete the parallelogram he , and draw the diagonal Ck , then will Ck represent the quantity and direction of the

pressure of the three stones A B, B C, C D upon the joint D. Make $Dm = Ck$, and proceed as before.

It is evident from this construction that the pressure of the materials in the vertical sections of the dome changes its direction continually : viz. from A B to B C, to C D, and so on ; and, if the breadths A B, B C, &c. of the stones were indefinitely small, the polygon A B, B C, C D, &c. would become a curve concave towards the axis of the dome.

Now, in forming a cylindrical vault which should support itself without any loading, the above construction might be employed, and in that case, the several verticals B c, C h, D o, &c. which represent the weights of the arch stones, or the vertical pressures which they exert upon the next lower joints C, D, E, &c. must be equal to each other when the lengths of the arch stones are equal, which is here supposed, and each must be equal to twice g v (g v representing the weight of a half voussoir) ; consequently the polygon formed would be an approximation to the common catenary. But in forming a dome, the keystone at A rests on a circular base formed by the next inferior horizontal course, and the lower circumference of each

horizontal course is greater than the upper. Therefore each horizontal course rests upon a greater number of points than the course next superior to it, and the number of points which the courses rest upon is proportional to the circumferences, or to the radii of their bases. Hence, if we have assumed any line, as $g v$, to represent the weight of the half arch-stone $A B$, or rather to represent the vertical pressure which that half stone exerts on the oblique joint B ; then it is plain that to get the weight of the arch-stone $B C$, or rather the vertical pressure it exerts against the oblique joint C , we must make $B c$ equal to $2 g v$ diminished in the proportion of the radius of the horizontal course $B C$ taken at C , to the radius of the same course, taken at B ; that is $z C : w B :: 2 g v : B c$. Afterwards $C h$ instead of being made equal to $B c$, must be obtained by the following proportion $x D : z C :: B c : C h$; and in the same manner the other verticals may be found.

It must be observed, however, that the radii $z C$, $x D$, &c. cannot be accurately known till the positions of the courses $B C$, $C D$, &c. are determined, but an approximation may be made to these radii by first making $B c = 2 g v$ determining the direction of the diagonal $B d$, and

the position of C as described above. The radius zC may then be employed to get a more correct value of Bc , from which the position of C may be determined with sufficient accuracy. In the same manner the approximate value of xD may be found, and from thence the more correct place of the point D, and so on.

Now, if the dome were so constructed that the several lines of direction AB , BC , &c. were always within the thickness of the voussoirs, it would stand and be in perfect equilibration; but if the lines of direction of the forces fall on the exterior of the curve of the dome, it is evident that the pressures of the upper courses would give the lower ones a greater tendency to slide outward at the joints than would be counteracted by their weights, and consequently the dome would fall.

Again, if the directions of the forces fell on the interior of the curve, the stones would tend to fall inward, but as this would take place equally in every stone of each horizontal course respectively, the tendency would not be obeyed, the breadth of the stones on the exterior being greater than on the interior. It is even considered that in proportion to the flatness of the dome, the action of gravity binds the stones in

the horizontal courses together more firmly. In this case, however, its action in the vertical planes increases the horizontal thrust of the dome towards the exterior, in the direction of the radii, and the flatness may be such as to cause the evil arising from the tendency in the latter direction, to exceed the advantage gained by the former. If, however, the foot of the dome is sufficiently secured against giving away by the horizontal thrust outward, then the dome may be considered more firm in proportion as the curve falls within that of equilibration; within this limit any curve, whether convex or concave, may be chosen for the vertical section of a dome.

Fig. 2, Plate 4, exhibits the curve of equilibration for one half of a dome, and is constructed from a table calculated by Dr. Robison.

It is evident that the voussoirs of domes being bound together by forces acting both in vertical and horizontal planes, form a building, having greater stability than a cylindrical vault, in which the voussoirs are only pressed together in vertical planes: perforation may be made in any part of a dome without sensible injury, the upper part may be either left quite open as in the Pantheon, or it may be crowned by a cupola, as in the cathedrals of St. Peter and St. Paul.

Dr. Robison shews that when a dome is spherical, it is not safe to employ a segment of more than about 103 degrees for the vertical section: beyond this limit the lower part would fall within the lines of direction of the forces.

TO DETERMINE THE HORIZONTAL THRUST OF A
DOME AT ITS BASE.

If the courses of which a dome is composed be indefinitely thin, or, at least, if their thickness bears but a small proportion to the magnitude of the dome, and if the directions of the pressures of the courses in a vertical plane be perpendicular to the joints, as in the dome of equilibration, then it is evident that the pressure upon the lowest joint all round the base of the dome will be the same as would be produced by a cone of equal weight, whose slant side is coincident with the direction of a tangent to the curve of the dome in a vertical plane at the springing course.

Let the curve $V A$, Fig. 3, Plate 4, be that by whose revolution about $V B$ the given dome is formed, and let $A C$ be a tangent to $V A$ at the

springing. Then, if the cone formed by the revolution of $A C$ be of equal weight with the dome, the pressure of the cone will be equal to that of the dome.

Now, let the whole weight of the dome or cone be accumulated in one point of the side of the cone, suppose at C ; then the weight will be to the horizontal thrust at A , produced by that weight, as $C B$ to $A B$.

If, for example, the whole weight of the dome should be 1000 tons, and that $A B$ should be equal to the half of $C B$, the horizontal thrust at A would be equal to 500 tons; but this thrust is distributed all round the base of the dome, and we are to find what it is equivalent to on every point of that base.

Since $A B$ may be taken to represent the thrust of the dome, we may consider a line equal in length to $A B$, as expressive of a force which would resist that thrust when applied at one point; and, we may consider the whole thrust as equally diffused over a line equal to $A B$. Consequently the pressure which each point in such a line would have to support, will diminish as the length of the line increases; that is, the pressure on any point would be inversely proportional to the length of the line. Therefore,

if we suppose the thrust of the dome to be resisted by a force acting all round the base, and tending every where towards the centre, the thrust upon each point of the line equal to $A B$, will be to that on each point of the base inversely, as that line is to the circumference of the base: that is inversely as the semi-diameter of a circle is to its circumference. But the radius being 1, the circumference is 6.283. Consequently, to find the horizontal thrust on every point of the base in the above example, we may say $6.283 : 1 :: 500 \text{ tons} : 79\frac{1}{2} \text{ tons}$, nearly the thrust required.

If the dome be supported on a circular wall of masonry, whose height is $A G$, the above thrust must be multiplied by the arm $A G$ of the lever, at the end of which it acts, and the product will express the force with which the dome endeavours to upset the wall. This, of course, must be resisted, as in the case of common vaulting, by a force expressed by the area of the section $A D$ of the wall, multiplied by the half breadth $D H$, (supposing the wall and dome constructed of materials having the same specific gravity, and the pier rectangular,) and as all the terms are given, except $D G$, this may also be found.

If the dome is intended to be hooped with iron at the base as usual, and we would give such dimensions to the ring that it shall resist the strain, we have only to find from the tables that are published, how much a bar of iron of given dimensions, (whose section is one square inch for example,) will support, without breaking, when a load is diffused uniformly over it; then the horizontal thrust divided by this tabular number, will give the number of square inches in a section of the intended ring.

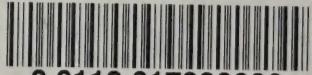
THE END.

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